NEW SUFFICIENT CONDITIONS FOR ANALYTIC AND UNIVALENT FUNCTIONS

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Abstract. The object of the present paper is to obtain new sufficient conditions on $f'''(z)$ which lead to some subclasses of univalent functions defined in the open unit disk.

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1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form :

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*(\alpha)$, the class of starlike functions of order α, $0 \le \alpha < 1$, if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathcal{U}). \tag{2}$$

Then $\mathcal{S}^* = \mathcal{S}^*(0)$ is the class of starlike functions in $\mathcal{U}$. Further, if $f(z) \in \mathcal{A}$ satisfies

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2}, \qquad (z \in \mathcal{U}) \tag{3}$$

for some $0 < \alpha \le 1$, then $f(z)$ said to be strongly starlike function of order α in $\mathcal{U}$, and this class denoted by $\overline{\mathcal{S}}^*(\alpha)$. Note that $\overline{\mathcal{S}}^*(1) = \mathcal{S}^*$.

Also let $\mathcal{K}(\alpha)$, $0 \le \alpha < 1$, which consists of functions $f(z) \in \mathcal{A}$ such that

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha \qquad (z\in\mathcal{U}), \tag{4}$$

and $\mathcal{K}=\mathcal{K}(0)$ is the class of convex functions in $\mathcal{U}$. Further, if $f(z)\in\mathcal{A}$ satisfies

$$\left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right|<\frac{\alpha\pi}{2}, \qquad (z\in\mathcal{U}) \tag{5}$$

for some $0<\alpha\le 1$, then $f(z)$ said to be strongly convex function of order α in $\mathcal{U}$, and this class denoted by $\overline{\mathcal{K}}(\alpha)$. Note that $\overline{\mathcal{K}}(1)=\mathcal{K}$ and if $zf'(z)\in\overline{\mathcal{S}}^*(\alpha)$ then $f(z)\in\overline{\mathcal{K}}(\alpha)$.

Furthermore, a function $f(z)\in\mathcal{A}$ is said to be in the class $\mathcal{R}(\alpha)$ if and only if

$$\mathrm{Re}\{f'(z)\}>\alpha \qquad (0\le\alpha<1;\ z\in\mathcal{U}), \tag{6}$$

Also, let $\overline{\mathcal{R}}(\alpha)$ be the subclasses of $\mathcal{A}$ which satisfies

$$|\arg f'(z)|<\frac{\alpha\pi}{2}, \qquad (z\in\mathcal{U}) \tag{7}$$

for some $0<\alpha\le 1$.

All the above mentioned classes are subclasses of univalent functions in $\mathcal{U}$.

There are many results for sufficient conditions of functions $f(z)$ which are analytic in $\mathcal{U}$ to be starlike, convex, strongly starlike and strongly convex functions have been given by several researchers. ([1], [2], [3], [4], [5], [6], [7], [8], [9]). In this paper we will study λ such that the condition$|f'''(z)|\le\lambda$, $z\in\mathcal{U}$, implies that $f(z)$ belongs to one of the classes defined above.

In order to prove our main result, we shall need the following lemmas obtained by Tuneski ([9]).

Lemma 1. *If* $f(z)\in\mathcal{A}$ *satisfies*

$$|f''(z)|\le\frac{2(1-\alpha)}{1-\alpha} \qquad (z\in\mathcal{U}; 0\le\alpha<1) \tag{8}$$

then $f(z)\in\mathcal{S}^*(\alpha)$.

Lemma 2. *If* $f(z)\in\mathcal{A}$ *satisfies*

$$|f''(z)|\le\frac{2\sin(\alpha\pi/2)}{1+\sin(\alpha\pi/2)} \qquad (z\in\mathcal{U}; 0<\alpha\le 1) \tag{9}$$

then $f(z) \in \overline{\mathcal{S}}^*(\alpha)$.

Lemma 3. *If* $f(z) \in \mathcal{A}$ *satisfies*

$$|f''(z)| \leq \frac{1-\alpha}{2-\alpha} \qquad (z \in \mathcal{U}; 0 \leq \alpha < 1) \tag{10}$$

then $f(z) \in \mathcal{K}(\alpha)$.

Lemma 4. *If* $f(z) \in \mathcal{A}$ *satisfies*

$$|f''(z)| \leq 1-\alpha \qquad (z \in \mathcal{U};\ 0 \leq \alpha < 1) \tag{11}$$

then $f(z) \in \mathcal{R}(\alpha)$.

Lemma 5. *If* $f(z) \in \mathcal{A}$ *satisfies*

$$|f''(z)| \leq \sin\frac{\alpha\pi}{2} \qquad (z \in \mathcal{U};\ 0 < \alpha \leq 1) \tag{12}$$

then $f(z) \in \overline{\mathcal{R}}(\alpha)$.

As a consequence of Theorem 3 in [10], we obtain the following lemma:

Lemma 6. *If* $f(z) \in \mathcal{A}$, *satisfies*

$$|f'(z) + \alpha z f''(z) - 1| < (1+\alpha)\sqrt{(\alpha-1)/(\alpha^2+3\alpha)} \qquad (z \in \mathcal{U}) \tag{13}$$

where $\alpha > 1$, *then* $f(z) \in \mathcal{K}(1/2)$.

2.Main Result

Theorem 1. *If* $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{A}$ *satisfies*

$$|f'''(z)| \leq \frac{2(1-\alpha)}{1-\alpha} - 2\,|a_2| \qquad (z \in \mathcal{U};\ 0 \leq \alpha < 1) \tag{14}$$

then $f(z) \in \mathcal{S}^*(\alpha)$.

Proof. Noting that

$$\begin{aligned} |f''(z)| - 2|a_2| &\leq |f''(z) - 2a_2| \\ &= \left| \int_0^z f'''(\sigma) d\sigma \right| \\ &\leq \int_0^{|z|} \left| f'''(te^{i\theta}) \right| dt \end{aligned}$$

and thus,

$$\begin{aligned} |f''(z)| &\leq \int_0^{|z|} \left| f'''(te^{i\theta}) \right| dt + 2|a_2| \\ &\leq \left(\frac{2(1-\alpha)}{1-\alpha} - 2|a_2| \right) \int_0^{|z|} dt + 2|a_2| \\ &\leq \left(\frac{2(1-\alpha)}{1-\alpha} - 2|a_2| \right) |z| + 2|a_2| \\ &\leq \frac{2(1-\alpha)}{1-\alpha}. \end{aligned}$$

Then by Lemma 1, we have $f(z) \in \mathcal{S}^*(\alpha)$.

Corollary 1. *Let* $f(z) \in \mathcal{A}$ *with* $f''(0) = 0$ *. Then*
(i) $|f'''(z)| \leq 2$ *implies* $f(z) \in \mathcal{S}^*$;
(ii) $|f'''(z)| \leq 4/5$ *implies* $f(z) \in \mathcal{S}^*(1/3)$;
(iii) $|f'''(z)| \leq 2/3$ *implies* $f(z) \in \mathcal{S}^*(1/2)$; *and*
(iv) $|f'''(z)| \leq 1/2$ *implies* $f(z) \in \mathcal{S}^*(2/3)$.

Applying the same method as in the proof of Theorem 1 and using Lemmas 2-5 instead of Lemma 1, we obtain the following theorems, respectively.

Theorem 2. *If* $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{A}$ *satisfies*

$$|f'''(z)| \leq \frac{2\sin(\alpha\pi/2)}{1+\sin(\alpha\pi/2)} - 2|a_2| \qquad (z \in \mathcal{U};\ 0 < \alpha \leq 1) \tag{15}$$

then $f(z) \in \overline{\mathcal{S}}^*(\alpha)$.

Corollary 2. *Let* $f(z) \in \mathcal{A}$ *with* $f''(0) = 0$ *. Then*
(i) $|f'''(z)| \leq 1$ *implies* $f(z) \in \mathcal{S}^*$;
(ii) $|f'''(z)| \leq 2/3$ *implies* $f(z) \in \overline{\mathcal{S}}^*(1/3)$;.
(iii) $|f'''(z)| \leq 2(\sqrt{2} - 1) = 0.8284\ldots$ *implies* $f(z) \in \overline{\mathcal{S}}^*(1/2)$; *and*
(iii) $|f'''(z)| \leq 2(2\sqrt{3} - 3) = 0.9282\ldots$ *implies* $f(z) \in \overline{\mathcal{S}}^*(2/3)$.

Theorem 3. *If* $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{A}$ *satisfies*

$$|f'''(z)| \leq \frac{1-\alpha}{2-\alpha} - 2\,|a_2| \qquad (z \in \mathcal{U};\ 0 \leq \alpha < 1) \tag{16}$$

then $f(z) \in \mathcal{K}(\alpha)$.

Corollary 3. *Let* $f(z) \in \mathcal{A}$ *with* $f''(0) = 0$ *. Then*
(i) $|f'''(z)| \leq 1/2$ *implies* $f(z) \in \mathcal{K}$;.
(ii) $|f'''(z)| \leq 2/5$ *implies* $f(z) \in \mathcal{K}(1/3)$;.
(iii) $|f'''(z)| \leq 1/3$ *implies* $f(z) \in \mathcal{K}(1/2)$; *and*
(iv) $|f'''(z)| \leq 1/4$ *implies* $f(z) \in \mathcal{K}(2/3)$.

Theorem 4. *If* $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{A}$ *satisfies*

$$|f'''(z)| \leq 1 - \alpha - 2\,|a_2| \qquad (z \in \mathcal{U};\ 0 \leq \alpha < 1) \tag{17}$$

then $f(z) \in \mathcal{R}(\alpha)$.

Corollary 4. *Let* $f(z) \in \mathcal{A}$ *with* $f''(0) = 0$ *. Then*
(i) $|f'''(z)| \leq 1$ *implies* $\operatorname{Re} f(z) > 0$;
(i) $|f'''(z)| \leq 2/3$ *implies* $f(z) \in \mathcal{R}(1/3)$;.
(ii) $|f'''(z)| \leq 1/2$ *implies* $f(z) \in \mathcal{R}(1/2)$; *and*
(iii) $|f'''(z)| \leq 1/3$ *implies* $f(z) \in \mathcal{R}(2/3)$.

Theorem 5. *If* $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{A}$ *satisfies*

$$|f'''(z)| \leq \sin\frac{\alpha\pi}{2} - 2\,|a_2| \qquad (z \in \mathcal{U};\ 0 < \alpha \leq 1) \tag{18}$$

then $f(z) \in \overline{\mathcal{R}}(\alpha)$.

Corollary 5. *Let* $f(z) \in \mathcal{A}$ *with* $f''(0) = 0$ *. Then*
(i) $|f'''(z)| \leq 1/2$ *implies* $f(z) \in \overline{\mathcal{R}}(1/3)$;
(ii) $|f'''(z)| \leq \sqrt{2}/2 = 0.7071\ldots$ *implies* $f(z) \in \overline{\mathcal{R}}(1/2)$; *and*
(iii) $|f'''(z)| \leq \sqrt{3}/2 = 0.8660\ldots$ *implies* $f(z) \in \overline{\mathcal{R}}(2/3)$.

Finally, we prove

Theorem 6. *If* $f(z) \in \mathcal{A}$, *satisfies*

$$|f''(z)| < \sqrt{(\alpha - 1)/(\alpha^2 + 3\alpha)} \qquad (z \in \mathcal{U}) \tag{19}$$

where $\alpha > 1,$ *then* $f(z) \in \mathcal{K}(1/2)$.

Proof. It follows that

$$\begin{aligned}
|f'(z) + \alpha z f''(z) - 1| &\leq |f'(z) - 1| + \alpha |zf''(z)| \\
&\leq \left| \int_0^z f''(t)dt \right| + \alpha |zf''(z)| \\
&\leq \int_0^{|z|} |f''(t)dt| + \alpha \sqrt{(\alpha - 1)/(\alpha^2 + 3\alpha)}\, |z| \\
&\leq (1 + \alpha)\sqrt{(\alpha - 1)/(\alpha^2 + 3\alpha)}\, |z| \\
&< (1 + \alpha)\sqrt{(\alpha - 1)/(\alpha^2 + 3\alpha)},
\end{aligned}$$

Using Lemma 6, we have $f(z) \in \mathcal{K}(1/2)$.

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