ON THE UNIVALENCE OF SOME INTEGRAL OPERATORS

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Abstract. In view of two integral operators $H_{\gamma_1,\gamma_2,...,\gamma_n}$ and $J_{\gamma_1,\gamma_2,...,\gamma_n}$ for analytic functions $f_j, j=\overline{1,n}$ in the open unit disk $\mathcal{U}$, sufficient conditions for univalence of these integral operators are discussed.

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1. Introduction

We consider the unit open disk $\mathcal{U}=\{z\in\mathbb{C}:|z|<1\}$ and $\mathcal{A}$ the class of the functions f which are analytic in $\mathcal{U}$ and $f(0)=f'(0)-1=0$.

We denote by $\mathcal{S}$ the subclass of the functions $f\in\mathcal{A}$ which are univalent in $\mathcal{U}$ and $\mathcal{S}^*$ denote the subclass of $\mathcal{S}$ consisting of all starlike functions f in $\mathcal{U}$.

We consider the integral operators

$$J_\gamma(z)=\left\{\frac{1}{\gamma}\int_0^z u^{-1}\left(f(u)\right)^{\frac{1}{\gamma}}du\right\}^\gamma \tag{1}$$

for $f\in\mathcal{A}$, γ be a complex number, $\gamma\neq 0$ and

$$H_{\gamma_1,\gamma_2,...,\gamma_n}(z)=\int_0^z\left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}}...\left(\frac{f_n(u)}{u}\right)^{\frac{1}{\gamma_n}}du \tag{2}$$

for $f_j\in\mathcal{A}$ and γ_j complex numbers, $\gamma_j\neq 0, j=\overline{1,n}$.

Miller and Mocanu [5] have studied that the integral operator J_γ is in the class $\mathcal{S}$ for $f\in\mathcal{S}^*$.

From (2) for $n = 1, f_1 = f$ and $\frac{1}{\gamma_1} = \alpha$ we obtain the integral operator Kim-Merkes, H_α, given by

$$H_\alpha(z) = \int_0^z \left(\frac{f(u)}{u}\right)^\alpha du \tag{3}$$

In [3] Kim-Merkes prove that the integral operator H_α is in the class $\mathcal{S}$ for $|\alpha| \leq \frac{1}{4}$ and $f \in \mathcal{S}$.

Pescar in [8], [9], has obtained univalence sufficient conditions for the integral operator H_α.

Pescar, Owa [12] have studied univalence problems for integral operator H_α.

In [2], D. Breaz and N. Breaz have studied the univalence of integral operator $H_{\gamma_1,\gamma_2,\ldots,\gamma_n}$.

In this paper we introduce a general integral operator

$$J_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z) = \left\{\left(\sum_{j=1}^n \frac{1}{\gamma_j}\right) \int_0^z u^{-1} f_1(u)^{\frac{1}{\gamma_1}} \ldots f_n(u)^{\frac{1}{\gamma_n}} du\right\}^{\frac{1}{\sum_{j=1}^n \frac{1}{\gamma_j}}} \tag{4}$$

for $f_j \in \mathcal{A}$, γ_j complex numbers, $\gamma_j \neq 0, j = \overline{1,n}$, which is a generalization of integral operator J_γ, given by (1).

For $n = 1, f_1 = f$ and $\gamma_1 = \gamma$, from (4) we obtain the integral operator J_γ.

In the present paper, we obtain some sufficient conditions for the integral operators $H_{\gamma_1,\gamma_2,\ldots,\gamma_n}$ and $J_{\gamma_1,\gamma_2,\ldots,\gamma_n}$ to be in the class $\mathcal{S}$.

2. Preliminary results

We need the following lemmas.

Lemma 2.1. [7] *Let α be a complex number, $Re\ \alpha > 0$ and $f \in \mathcal{A}$. If*

$$\frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} \left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{5}$$

for all $z \in \mathcal{U}$, then the integral operator F_α defined by

$$F_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} f'(u) du\right]^{\frac{1}{\alpha}} \tag{6}$$

is in the class S.

Lemma 2.2. [1] *If* $f(z) = z + a_2 z^2 + ..$ *is analytic in* $\mathcal{U}$ *and*

$$(1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \tag{7}$$

for all $z \in \mathcal{U}$, *then the function* $f(z)$ *is univalent in* $\mathcal{U}$.

Lemma 2.3. (Schwarz [4]). *Let* f *the function regular in the disk* $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ *with* $|f(z)| < M$, M *fixed. If* $f(z)$ *has in* $z = 0$ *one zero with multiply* $\geq m$, *then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m, \; z \in \mathcal{U}_R \tag{8}$$

the equality (in the inequality (8) for $z \neq 0$*) can hold only if*

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ *is constant.*

3. Main results

Theorem 3.1 *Let* γ_j *be complex numbers,* $\gamma_j \neq 0$, M_j *real positive numbers,* $j = \overline{1,n}$, $\sum_{j=1}^{n} Re \frac{1}{\gamma_j} = 1$ *and* $f_j \in \mathcal{A}$, $f_j(z) = z + a_{2j} z^2 + a_{3j} z^3 + ..., j = \overline{1,n}$. *If*

$$\left| \frac{zf_j'(z)}{f_j(z)} - 1 \right| \leq M_j, \; j = \overline{1,n} \tag{9}$$

and

$$\sum_{j=1}^{n} \frac{M_j}{|\gamma_j|} \leq \frac{3\sqrt{3}}{2} \tag{10}$$

then the integral operators $J_{\gamma_1, \gamma_2, ..., \gamma_n}$ *given by (4) and* $H_{\gamma_1, \gamma_2, ..., \gamma_n}$ *given by (2) are in the class* $\mathcal{S}$.

Proof. We observe that

$$J_{\gamma_1, \gamma_2, ..., \gamma_n}(z) =$$

$$= \left\{ \left(\sum_{j=1}^{n} \frac{1}{\gamma_j} \right) \int_0^z u^{\sum_{j=1}^{n} \frac{1}{\gamma_j} - 1} \left(\frac{f_1(u)}{u} \right)^{\frac{1}{\gamma_1}} \dots \left(\frac{f_n(u)}{u} \right)^{\frac{1}{\gamma_n}} du \right\}^{\frac{1}{\sum_{j=1}^{n} \frac{1}{\gamma_j}}} \quad (11)$$

Let us consider the function

$$H_{\gamma_1,\gamma_2,\dots,\gamma_n}(z) = \int_0^z \left(\frac{f_1(u)}{u} \right)^{\frac{1}{\gamma_1}} \dots \left(\frac{f_n(u)}{u} \right)^{\frac{1}{\gamma_n}} du \quad (12)$$

for $f_j \in \mathcal{A}$, $j = \overline{1,n}$. The function $H_{\gamma_1,\gamma_2,\dots,\gamma_n}(z)$ is regular in $\mathcal{U}$.

We define the function p by $p(z) = \frac{zH''_{\gamma_1,\gamma_2,\dots,\gamma_n}(z)}{H'_{\gamma_1,\gamma_2,\dots,\gamma_n}(z)}$. The function p satisfies $p(0) = 0$ and

$$|p(z)| \le \sum_{j=1}^{n} \left(\frac{1}{|\gamma_j|} \left| \frac{zf'_j(z)}{f_j(z)} - 1 \right| \right) , z \in \mathcal{U} \quad (13)$$

From (9) and (13) we have

$$|p(z)| \le \sum_{j=1}^{n} \frac{M_j}{|\gamma_j|} \quad (14)$$

for all $z \in \mathcal{U}$.

Applying Lemma 2.3 we obtain

$$|p(z)| \le \sum_{j=1}^{n} \frac{M_j}{|\gamma_j|} |z| , z \in \mathcal{U} \quad (15)$$

and hence, we get

$$\left(1 - |z|^2\right) \left| \frac{zH''_{\gamma_1,\gamma_2,\dots,\gamma_n}(z)}{H'_{\gamma_1,\gamma_2,\dots,\gamma_n}(z)} \right| \le (1 - |z|^2)|z| \sum_{j=1}^{n} \frac{M_j}{|\gamma_j|} \quad (16)$$

for all $z \in \mathcal{U}$.

We have

$$\max_{|z| \le 1} \left\{ (1 - |z|^2)|z| \right\} = \frac{2}{3\sqrt{3}}$$

and from (10) and (16) we obtain

$$\left(1 - |z|^2\right) \left| \frac{zH''_{\gamma_1,\gamma_2,\dots,\gamma_n}(z)}{H'_{\gamma_1,\gamma_2,\dots,\gamma_n}(z)} \right| \le 1 \quad (17)$$

for all $z \in \mathcal{U}$.

From (17) and by Lemma 2.2, we obtain that the integral operator $H_{\gamma_1,\gamma_2,\ldots,\gamma_n}$ is in the class $\mathcal{S}$.

Because $H'_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z) = \left(\frac{f_1(z)}{z}\right)^{\frac{1}{\gamma_1}} \ldots \left(\frac{f_n(z)}{z}\right)^{\frac{1}{\gamma_n}}$ and

$$Re\ \alpha = \sum_{j=1}^{n} Re\frac{1}{\gamma_j} = 1,$$

from (17) and by Lemma 2.1 it results that the integral operator $J_{\gamma_1,\gamma_2,\ldots,\gamma_n}$ belongs to class $\mathcal{S}$. □

Corollary 3.2 *Let γ be a complex number, $Re\frac{1}{\gamma} = 1$ and $f \in \mathcal{A}$,*

$$f(z) = z + a_{21}z^2 + a_{31}z^3 + \ldots$$

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{3\sqrt{3}}{2}|\gamma| \tag{18}$$

for all $z \in \mathcal{U}$, then the integral operators J_γ and $H_\alpha, \alpha = \frac{1}{\gamma}$, are in the class $\mathcal{S}$.

Proof. From Theorem 3.1, for $n = 1, \gamma_1 = \gamma, \alpha = \frac{1}{\gamma}, f_1 = f$ it results that $J_\gamma \in \mathcal{S}$ and $H_\alpha \in \mathcal{S}$. □

Corollary 3.3. *If $f \in \mathcal{A}$ and*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{3\sqrt{3}}{2},\ z \in \mathcal{U} \tag{19}$$

then the integral operator of Alexander H given by

$$H(z) = \int_0^z \frac{f(u)}{u} du \tag{20}$$

is in the class $\mathcal{S}$.

Proof. From Theorem Theorem 3.1, for $n = 1, \gamma_1 = 1, f_1 = f, H = H_1$ we obtain $H \in \mathcal{S}$. □

Theorem 3.4. *Let γ_j be complex numbers and $f_j \in \mathcal{S}$, $f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \ldots$, $j = \overline{1.n}$.*

If

$$\sum_{j=1}^{n} \frac{1}{|\gamma_j|} \leq \frac{1}{4} \tag{21}$$

then the integral operator $H_{\gamma_1,\gamma_2,\ldots,\gamma_n}$ *given by (2) is in the class* $\mathcal{S}$.

Proof. We consider the function $p(z) = \frac{zH''_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z)}{H'_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z)}$, where $H_{\gamma_1,\gamma_2,\ldots,\gamma_n}$ is defined by (2). We obtain

$$|p(z)| \leq \sum_{j=1}^{n} \frac{1}{|\gamma_j|} \left| \frac{zf'_j(z)}{f_j(z)} - 1 \right|, \; z \in \mathcal{U} \tag{22}$$

Because $f_j \in \mathcal{S}$ we have $\left| \frac{zf'_j(z)}{f_j(z)} \right| \leq \frac{1+|z|}{1-|z|}, \; z \in \mathcal{U}, \; j = \overline{1,n}$ and

$$\left| \frac{zf'_j(z)}{f_j(z)} - 1 \right| \leq \left| \frac{zf'_j(z)}{f_j(z)} \right| + 1 \leq \frac{2}{1-|z|}, \; j = \overline{1,n}, \; z \in \mathcal{U} \tag{23}$$

From (21), (23) and (22) we obtain

$$|p(z)| \leq \frac{1}{2(1-|z|)}, \; z \in \mathcal{U} \tag{24}$$

and hence, we have

$$(1-|z|^2) \left| \frac{zH''_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z)}{H'_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z)} \right| \leq 1 \tag{25}$$

for all $z \in \mathcal{U}$.

By Lemma 2.2 we have $H_{\gamma_1,\gamma_2,\ldots,\gamma_n}$ is in the class $\mathcal{S}$. □

Corollary 3.5. *Let* γ *be a complex number and* $f \in \mathcal{S}$, $f = z + a_{21}z^2 + \ldots$ *If*

$$|\gamma| \leq \frac{1}{4} \tag{26}$$

then the integral operator $H_\gamma \in \mathcal{S}$.

Proof. In Theorem Theorem 3.4 for $n = 1$, $\frac{1}{\gamma_1} = \gamma$, $f_1 = f$ it results that H_γ is in the class $\mathcal{S}$. □

Theorem 3.6. *Let γ_j be complex numbers, $\gamma_j \neq 0$, $f_j \in \mathcal{A}$, $f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \dots$, $j = \overline{1,n}$ and $a = \sum_{j=1}^{n} Re\, \frac{1}{\gamma_j} > 0$. If*

$$\left| \frac{zf_j'(z)}{f_j(z)} - 1 \right| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2n} |\gamma_j|,\ j = \overline{1,n} \tag{27}$$

then the integral operator $J_{\gamma_1,\gamma_2,\dots,\gamma_n}$ is in the class $\mathcal{S}$.

Proof. The integral operator $J_{\gamma_1,\gamma_2,\dots,\gamma_n}$ has the form (11).

We consider the function

$$p(z) = \frac{zH''_{\gamma_1,\gamma_2,\dots,\gamma_n}(z)}{H'_{\gamma_1,\gamma_2,\dots,\gamma_n}(z)},\ z \in \mathcal{U} \tag{28}$$

where $H_{\gamma_1,\gamma_2,\dots,\gamma_n}(z)$ is define by (2).

The function p satisfies $p(0) = 0$ and from (28) we obtain

$$|p(z)| \leq \sum_{j=1}^{n} \left(\frac{1}{|\gamma_j|} \left| \frac{zf_j'(z)}{f_j(z)} - 1 \right| \right),\ z \in U \tag{29}$$

From (27) and (29) we have

$$|p(z)| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2} \tag{30}$$

for all $z \in \mathcal{U}$.

Applying Lemma 2.3 we obtain

$$|p(z)| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2} |z|,\ z \in \mathcal{U} \tag{31}$$

From (28) and (31) we have

$$\frac{1-|z|^{2a}}{a} \left| \frac{zH''_{\gamma_1,\gamma_2,\dots,\gamma_n}(z)}{H'_{\gamma_1,\gamma_2,\dots,\gamma_n}(z)} \right| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2} \frac{1-|z|^{2a}}{a} |z| \tag{32}$$

for all $z \in \mathcal{U}$.

Because

$$\max_{|z|\leq 1} \left\{ \frac{1-|z|^{2a}}{a} |z| \right\} = \frac{2}{(2a+1)^{\frac{2a+1}{2a}}}\ ,$$

from (32) we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zH''_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z)}{H'_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z)}\right| \leq 1 \tag{33}$$

for all $z \in \mathcal{U}$.

From (2) we have

$$H'_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z) = \left(\frac{f_1(z)}{z}\right)^{\frac{1}{\gamma_1}} \cdots \left(\frac{f_n(z)}{z}\right)^{\frac{1}{\gamma_n}},\ z \in \mathcal{U} \tag{34}$$

and from (33) by Lemma 2.1 it results that the integral operator $J_{\gamma_1,\gamma_2,\ldots,\gamma_n}$ belongs to class $\mathcal{S}$. □

Remark 3.7. *From Theorem 3.6 for* $n = 1, \gamma_1 = \gamma, f_1 = f, a = Re\,\frac{1}{\gamma} = 1$ *we obtain the condition*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{3\sqrt{3}}{2}|\gamma| \tag{35}$$

and, hence, we obtain Corollary 3.2.

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