UNIVALENCE CRITERIA

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Abstract. In this paper we derive some criteria for univalence of analytic functions and of an integral operator in the open unit disk.

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1. Introduction

Let $\mathcal{A}$ be the class of the functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let S denote the subclass of $\mathcal{A}$ consisting of all univalent functions f in $\mathcal{U}$.

2. Preliminary results

We need the following theorems.

Theorem 2.1.[1].*If $f(z) = z + a_2 z^2 + \dots$ is analytic in $\mathcal{U}$ and*

$$\left(1 - |z|^2\right) \left| \frac{z f''(z)}{f'(z)} \right| \leq 1 \tag{2.1}$$

for all $z \in \mathcal{U}$, then the function $f(z)$ is univalent in $\mathcal{U}$.

Theorem 2.2.[4]. *Let α be a complex number, $Re\,\alpha > 0$ and $f \in \mathcal{A}$. If*

$$\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{2.2}$$

for all $z \in \mathcal{U}$, then the function

$$F_\alpha(z) = \left[\alpha\int_0^z u^{\alpha-1}f'(u)du\right]^{\frac{1}{\alpha}} \tag{2.3}$$

is in the class $\mathcal{S}$.

Theorem 2.3.[5]*Let α be a complex number, $Re\,\alpha > 0$ and $f \in \mathcal{A}$. If*

$$\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1$$

for all $z \in \mathcal{U}$, then for any complex number β, $Re\,\beta \geq Re\,\alpha$, the function

$$F_\beta(z) = \left[\beta\int_0^z u^{\beta-1}f'(u)du\right]^{\frac{1}{\beta}} \tag{2.4}$$

is regular and univalent in $\mathcal{U}$.

Theorem 2.4.*(Schwarz)*[2].*Let $f(z)$ the function regular in the disk*

$$\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$$

with $|f(z)| < M$, M fixed. If $f(z)$ has in $z = 0$ one zero with multiply $\geq m$, then

$$|f(z)| \leq \frac{M}{R^m}|z|^m, \quad z \in \mathcal{U}_R \tag{2.5}$$

the equality (in the inequality (2.5) for $z \neq 0$) can hold only if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,$$

where θ is constant.

Theorem 2.5.[3]*If the function $g(z)$ is regular in $\mathcal{U}$ and $|g(z)| < 1$ in $\mathcal{U}$, then for all $\xi \in \mathcal{U}$ and $z \in \mathcal{U}$ the following inequalities hold:*

$$\left|\frac{g(\xi)-g(z)}{1-\overline{g(z)}g(\xi)}\right| \leq \left|\frac{\xi - z}{1-\overline{z}\xi}\right| \tag{2.6}$$

and

$$|g'(z)| \leq \frac{1-|g(z)|^2}{1-|z|^2} \tag{2.7}$$

the equalities hold only in the case $g(z) = \frac{\mathcal{E}(z+u)}{1+\bar{u}z}$, *where* $|\mathcal{E}| = 1$ *and* $|u| < 1$.

Remark.[3]For $z = 0$, from inequality (2.6) we have

$$\left| \frac{g(\xi) - g(0)}{1 - \overline{g(0)}g(\xi)} \right| < |\xi| \tag{2.8}$$

and, hence

$$|g(\xi)| \leq \frac{|\xi| + |g(0)|}{1 + |g(0)||\xi|} \tag{2.9}$$

Considering $g(0) = a$ and $\xi = z$,

$$|g(z)| \leq \frac{|z| + |a|}{1 + |a||z|} \tag{2.10}$$

for all $z \in \mathcal{U}$.

3. Main results

Theorem 3.1.*Let the function* $f \in \mathcal{A}$. *If*

$$\left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{3\sqrt{3}}{2} \tag{3.1}$$

for all $z \in \mathcal{U}$, *then the function* f *is in the class* $\mathcal{S}$.

Proof. We consider the function $g(z) = \frac{zf''(z)}{f'(z)}$, $z \in \mathcal{U}$. We have $g(0) = 0$ and from (3.1) by Theorem 2.4 (Schwarz) we obtain

$$\left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{3\sqrt{3}}{2}|z| \tag{3.2}$$

for all $z \in \mathcal{U}$. From (3.2) we get

$$(1-|z|^2)\left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{3\sqrt{3}}{2}(1-|z|^2)|z| \tag{3.3}$$

for all $z \in \mathcal{U}$.

Because

$$\max_{|z|<1}[(1-|z|^2)|z|] = \frac{2}{3\sqrt{3}},$$

from (3.3) we obtain

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{3.4}$$

for all $z \in \mathcal{U}$ and by Theorem 2.1 we obtain that f is in the class $\mathcal{S}$.

Theorem 3.2. *Let α be a complex number, $Re\ \alpha > 0$ and the function $f \in \mathcal{A}$. If*

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{(2Re\ \alpha+1)^{\frac{2Re\ \alpha+1}{Re\ \alpha}}}{2} \tag{3.5}$$

for all $z \in \mathcal{U}$, then the function

$$F_\alpha(z) = \left[\alpha\int_0^z u^{\alpha-1}f'(u)du\right]^{\frac{1}{\alpha}} \tag{3.6}$$

is regular and univalent in $\mathcal{U}$.

Proof. Let's consider the function $p(z) = \frac{zf''(z)}{f'(z)}$, $z \in \mathcal{U}$. We have $p(0) = 0$ and from (3.5) by Theorem 2.4 (Schwarz) we get

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{(2Re\ \alpha+1)^{\frac{2Re\ \alpha+1}{Re\ \alpha}}}{2}|z| \tag{3.7}$$

for all $z \in \mathcal{U}$. From (3.7) we obtain

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{(2Re\ \alpha+1)^{\frac{2Re\ \alpha+1}{Re\ \alpha}}}{2}\cdot\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z| \tag{3.8}$$

for all $z \in \mathcal{U}$.

Since

$$\max_{|z|<1}\left(\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}|z|\right) = \frac{2}{(2Re\ \alpha+1)^{\frac{2Re\ \alpha+1}{Re\ \alpha}}}$$

from (3.8) we have

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{3.9}$$

for all $z \in \mathcal{U}$ and by Theorem 2.2 we obtain that the function $F_\alpha(z)$ is regular and univalent in $\mathcal{U}$.

Remark 3.3. From Theorem 3.2 for $\alpha = 1$ we obtain Theorem 3.1.

Theorem 3.4. *Let α be a complex number, $Re\ \alpha > 0$ and the function $f \in \mathcal{A}$. If*

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{(2Re\ \alpha + 1)^{\frac{2Re\ \alpha+1}{Re\ \alpha}}}{2} \tag{3.10}$$

for all $z \in \mathcal{U}$, then for any complex number β, $Re\ \beta \geq Re\ \alpha$, the function

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u)du\right]^{\frac{1}{\beta}} \tag{3.11}$$

is regular and univalent in $\mathcal{U}$.

Proof. We consider the function $\psi : (0, \infty) \to \mathbb{R}$, $\psi(x) = \frac{1-a^{2x}}{x}$, $0 < a < 1$. The function $\psi(x)$ is the function decreasing for $x \in (0,1)$. If $x_1 = Re\ \alpha \leq x_2 = Re\ \beta$ and $a = |z|$, $z \in \mathcal{U}$ then

$$\frac{1 - |z|^{2Re\ \beta}}{Re\ \beta} \leq \frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} \tag{3.12}$$

for all $z \in \mathcal{U}$. From (3.12) we obtain

$$\frac{1 - |z|^{2Re\ \beta}}{Re\ \beta}\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \tag{3.13}$$

for all $z \in \mathcal{U}$.

From (3.10) and Theorem 2.4 (Schwarz) we get

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{(2Re\ \alpha + 1)^{\frac{2Re\ \alpha+1}{Re\ \alpha}}}{2}|z|, \quad z \in \mathcal{U} \tag{3.14}$$

and, hence, we have

$$\frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{(2Re\ \alpha + 1)^{\frac{2Re\ \alpha+1}{Re\ \alpha}}}{2} \cdot \frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha}|z| \tag{3.15}$$

for all $z \in \mathcal{U}$.

Because

$$\max_{|z|<1} \frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha}|z| = \frac{2}{(2Re\ \alpha + 1)^{\frac{2Re\ \alpha+1}{Re\ \alpha}}} \tag{3.16}$$

by (3.15) and (3.13) we obtain

$$\frac{1-|z|^{2Re\,\beta}}{Re\,\beta}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{3.17}$$

for all $z \in \mathcal{U}$.

From (3.17) and Theorem 2.2 we obtain that the function $F_\beta(z)$ is regular and univalent in $\mathcal{U}$.

Theorem 3.5. *Let α, β complex numbers $Re\,\beta \geq Re\,\alpha > 0$, the function $f \in \mathcal{A}, f(z) = z + a_2z^2 + \ldots$ If*

$$\left|\frac{f''(z)}{f'(z)}\right| < 1 \tag{3.18}$$

for all $z \in \mathcal{U}$ and

$$\max_{|z|<1}\left[\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}|z|\frac{|z|+2|a_2|}{1+2|a_2||z|}\right] \leq 1 \tag{3.19}$$

then the function $F_\beta(z)$ define by (2.4) is regular and univalent in $\mathcal{U}$.

Proof. Let's consider the regular function $p(z) = \frac{f''(z)}{f'(z)},\ z \in \mathcal{U}$. We have $|p(0)| = 2|a_2|$ and from (3.18) we obtain $|p(z)| < 1$ for all $z \in \mathcal{U}$.

By Remark 2.6 we obtain

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{|z|+2|a_2|}{1+2|a_2||z|} \tag{3.20}$$

for all $z \in \mathcal{U}$. From (3.20) we get

$$\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}|z|\frac{|z|+2|a_2|}{1+2|a_2||z|} \tag{3.21}$$

for all $z \in \mathcal{U}$.

We consider the function $Q : [0,1] \to \mathbb{R}$

$$Q(x) = \frac{1-x^{2Re\,\alpha}}{Re\,\alpha}x\frac{x+2|a_2|}{1+2|a_2|x};\quad x = |z|.$$

Because $Q\left(\frac{1}{2}\right) > 0$ it results that

$$\max_{x\in(0,1)} Q(x) > 0.$$

Using this result and from (3.21) we conclude

$$\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq \max_{|z|<1}\left[\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}|z|\frac{|z|+2|a_2|}{1+2|a_2||z|}\right] \tag{3.22}$$

and hence, by (3.19) we obtain

$$\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{3.23}$$

for all $z \in \mathcal{U}$. From (3.23) and by Theorem 2.3 we obtain that the function $F_\beta(z)$ is in the class $\mathcal{S}$.

References

[1] Becker, J., *Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Functionen*, J. Reine Angew. Math. 255 (1972), 23-43.

[2] Mayer, O., *The Functions Theory of One Variable Complex*, Bucureşti, 1981.

[3] Nehari, Z., *Conformal Mapping*, Mc Graw-Hill Book Comp., New York, 1952 (Dover. Publ. Inc., 1975).

[4] Pascu, N. N., *On a univalence criterion II*, Itinerant Seminar on Functional Equations, Approximation and Convexity (Cluj-Napoca, 1985), 153-154.

[5] Pascu, N. N., *An improvement of Becker's univalence criterion*, Proceedings of the Commemorative Session Simion Stoilow, Braşov, (1987), 43-48.

[6] Pescar, V., *New Univalence Criteria*, "Transilvania" University of Braşov, 2002.

[7] Pescar, V., Breaz, D. V., *The Univalence of Integral Operators*, Academic Publishing House, Sofia, 2008.

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