G-N-QUASIGROUPS

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Abstract. In this paper we present criteria for an n-quasigroup to be isotopic to an n-group. We call a such n-quasigroup $G-n$-quasigroup. Applications to functional equations on quasigroups are presented in a subsequent paper.

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Some important n-quasigroup classes are the following. An n-quasigroup (A, α) of the form $\alpha(x_1^n) = \sum_{i=1}^{n} f_i(x_i)+a$, where $(A, +)$ is a group, $f_1, \ldots, f_n$ are some automorphisms of $(A, +)$, a is some fixed element of A is called **linear n-quasigroup** (over group $(A, +)$). A linear quasigroup over an abelian group is called **$T - n$-quasigroup**. An n-quasigroup with identity

$$\alpha(\alpha(x_{11}^{1n}), \ldots, \alpha(x_{n1}^{nn})) = \alpha(\alpha(x_{11}^{n1}), \ldots, \alpha(x_{1n}^{nn})$$

is called **medial n-quasigroup**.

All these quasigroups are isotopic to n-groups. This motivates the purpose of our work to find criteria for an n-quasigroup to be isotopic to an n-group.

1. Preliminaries

Recall several notions and results which will be used in what follows.

A non-empty set A together with one n-ary operation $\alpha : A^n \to A$, $n > 2$ is called n-groupoid and is denoted by (A, α).

We shall use the following abbreviated notation:

- the sequence $x_i, \ldots, x_j$ will be denoted by x_i^j. For $j < i$, x_i^j is the empty symbol;

- if $x_{i+1} = \ldots = x_{i+k} = x$ then instead x_{i+1}^{i+k} we will write $(x)^k$. For $k \leq 0$, $(x)^k$ is the empty symbol.

(A, α) is an n-**semigroup** if α is associative, i.e.

$$\alpha(\alpha(x_1^n), x_{n+1}^{2n-1}) = \alpha(x_1, \alpha(x_2^{n+1}), x_{n+2}^{2n-1}) = \ldots = \alpha(x_1^{n-1}, \alpha(x_n^{2n-1}))$$

holds for all $x_1, \ldots, x_{2n-1} \in A$.

An element $e \in A$ is called an i-**unit** if $\alpha((e)^{i-1}, x, (e)^{n-1}) = x$ for all $x \in A$. If e is an i-unit for all $i = 1, 2, \ldots, n$ it is called an **unit**.

If each equation $\alpha(a_1^{i-1}, x, a_{i+1}^n) = b$ is uniquely solvable with respect to x, $i = 1, 2, \ldots, n$ for all $a_1, \ldots, a_n$, $b \in A$, (A, α) is called n-**quasigroup**. An n-quasigroup which has at least one unit is called n-**loop**.

We introduced in [4] the notion of homotopy of universal algebras. In particular, for n-groupoids we have the following. Let $\mathcal{A} = (A, \alpha)$ and $\mathcal{B} = (B, \beta)$ be n-groupoids. An ordered system of mappings $[f_1^n; f]$ from A to B such that $f(\alpha(a_1^n)) = \beta(f_1(a_1), \ldots, f_n(a_n))$ for all $a_1^n \in A^n$ is called a **homotopy** from $\mathcal{A}$ to $\mathcal{B}$. Equality and composition of homotopies are defined componentwise. Composition of homotopies produces a homotopy and is associative. An **isotopy** is a homotopy with all components bijections.

In many applications of quasigroups isotopies and homotopies are more important than isomorphisms and homomorphisms.

Any n-quasigroup is isotopic to an n-loop (see [1]).

Let (A, α) be an n-quasigroup and $a = a_1^n \in A^n$. The mapping $T_i : A \to A$, $T_i(x) = \alpha(a_1^{i-1}, x, a_{i+1}^n)$ is called the i-**th translation** by a, $i = 1, 2, \ldots, n$. Let $\overline{\alpha} : A^n \to A$ defined by $\overline{\alpha}(x_1^n) = \alpha(T_1^{-1}(x_1), \ldots, T_n^{-1}(x_n))$. Then $(A, \overline{\alpha})$ is an n-loop ($e = \alpha(a_1^n)$ is a unit) and $[T_1^n; 1_A]$ is an isotopy from (A, α) to $(A, \overline{\alpha})$.

$(A, \overline{\alpha})$ is called a LP-**isotope** of (A, α) and $[T_1^n; 1_A]$ a LP-**isotopy**.

In [4] we proved the following. Let (A, α) and (B, β) be n-quasigroups and $[f_1^n; f] : (A, \alpha) \to (B, \beta)$ a homotopy (isotopy), $a = a_1^n \in A^n$, $b = b_1^n \in B$, $b_i = f_i(a_i)$, T_i translations by a and U_i translations by b, $i = 1, 2, \ldots, n$. Then the following diagram is commutative and f is a homomorphism (isomorphism).

$$\begin{array}{ccc}
(A,\alpha) & \xrightarrow{[f_1^n; f]} & (B,\beta) \\
{\scriptstyle [T_1^n; 1_A]}\downarrow & & \downarrow{\scriptstyle [U_1^n; 1_B]} \\
(A,\overline{\alpha}) & \xrightarrow{f} & (B,\overline{\beta})
\end{array}$$

If (B,β) is an n-loop and $[f_1^n; f]$ an isotopy, choosing a_i such that $f_i(a_i) = u$, u a unit in (B,β) we obtain that $f : (A,\overline{\alpha}) \to (B,\beta)$ is an isomorphism.

An n-semigroup which is also an n-quasigroup is called n-**group** (see [2]).

Let (A,α) be an n-group. By Hosszu theorem (see [5]), $\alpha(x_1^n) = x_1 \cdot f(x_2) \cdot f^2(x_3) \cdot \ldots \cdot f^{n-1}(x_n) \cdot u$, where $(A,\cdot)$ is a binary group (called a creating group), $x \cdot y = \alpha(x,(a)^{n-2},y)$, $a \in A$ a fixed element, $f(x) = \alpha(\overline{a},x,(a)^{n-2})$ an automorphism of $(A,\cdot)$ and $u = \alpha((\overline{a})^n)$, $\overline{a}$ the skew element to a. Based on this result Belousov [1] proved that every LP-isotope (A,β) of (A,α) is an n-group derived from a binary group $(A,\circ)$, i.e. $\beta(x_1^n) = x_1 \circ x_2 \circ \ldots \circ x_n$ where $x \circ y = xe^{-1}y$, $e = \alpha(a_1^n)$.

We introduce the following.

Definition 1. *An n-quasigroup is called $G-n$-**quasigroup** (or shortly G-**quasigroup**) if it is isotopic to an n-group.*

From the above results if follows.

Theorem 1. *Every n-loop isotopic to a G-quasigroup is an n-group derived from a binary group.*

An n-group can also be defined as an algebra $(A,\alpha,-)$, $\alpha : A^n \to A$, $- : A \to A$ such that α is associative and the following identities are satisfied: $\alpha(\overline{x},(x)^{n-2},y) = y$, $\alpha(y,(x)^{n-2},\overline{x}) = y$ (see [2]). In [2] was proved that $(A,\alpha,-)$ is an abelian algebra (in the sense of general algebras - see [3]) iff α is semicommutative, i.e. $\alpha(x_1,x_2^{n-1},x_n) = \alpha(x_n,x_2^{n-1},x_1)$.

Definition 2. *An n-quasigroup is called G_a-n-**quasigroup** (shortly G_a-**quasigroup**) if it is isotopic to an abelian n-group.*

Theorem 2. *Every n-loop isotopic to a G_a-quasigroup is an n-group derived from a binary commutative group.*

Proof. We proved in [5] that an n-group (A, α) is abelian iff any of its creating groups $(A, \cdot)$ (see above) is commutative. Therefore every LP-isotope (A, β) of (A, α) is derived from a commutative binary group. Indeed, $(A, \circ)$ is isomorphic to $(A, \cdot)$, $h(x \circ y) = h(x)h(y)$, $h(x) = xe^{-1}$.

2. $G - n$-quasigroups

In this section we present criteria for an n-quasigroup to be isotopic to an n-group. We finish this section showing that G-3-quasigroups are connected with the functional equation of generalized associativity.

Let (A, α) be an n-quasigroup.

Definition 3. (see [1]). *We say that in (A, α)* ***condition*** *$D_{i,j}$, $1 \leq i < j \leq n$, holds if $\alpha(a_1^{i-1}, u_i^j, a_{j+1}^n) = \alpha(a_1^{i-1}, v_i^j, a_{j+1}^n)$ implies $\alpha(x_1^{i-1}, u_i^j, x_{j+1}^n) = \alpha(x_1^{i-1}, v_i^j, x_{j+1}^n)$ for all $x_1^n \in A^n$.*

It is obvious that condition $D_{i,j}$ is isotopic invariant.

Theorem 3. (see [1]). I*n (A, α) condition $D_{i,j}$ holds iff there exists quasigroups (A, β) of arity $j - i + 1$ and (A, γ) of arity $n - j + i$ such that $\alpha(x_1^n) = \gamma(x_1^{i-1}, \beta(x_i^j), x_{j+1}^n)$.*

Proof. Choose $a_i, \ldots, a_{j-1} \in A$. For any $x_1^n \in A^n$ there exists only one $b \in A$ such that $\alpha(x_1^n) = \alpha(x_1^{i-1}, a_i^{j-1}, b, x_{j+1}^n)$. By condition $D_{i,j}$ we obtain $\alpha(y_1^{i-1}, x_i^j, y_{j+1}^n) = \alpha(y_1^{i-1}, a_i^{j-1}, b, y_{j+1}^n)$, i.e. b depends only of $x_i^j : b = \beta(x_i^j)$, $\beta : A^{j-i+1} \to A$. Therefore we have $\alpha(x_1^n) = \alpha(x_1^{i-1}, a_i^{j-1}, \beta(x_i^j), x_{j+1}^n) = \gamma(x_1^{i-1}, \beta(x_i^j), x_{j+1}^n)$, where $\gamma(x_1^{n-j+i}) = \alpha(x_1^{i-1}, a_i^{j-1}, x_i^{n-j+i})$ is a retract of α. It is easy to prove that (A, β) and (A, γ) are quasigroups.

The converse is trivial.

We focus on conditions $D_{i,i+1}$.

Theorem 4. *Condition $D_{i,i+1}$ holds in (A, α) iff for each unit e of any n-loop (B, β) isotopic to (A, α) $\beta(x_1^{i-1}, e, x_i^{n-1}) = \beta(x_1^i, e, x_{i+1}^{n-1})$ for all $x_1^n \in B^n$.*

Proof. Suppose that in (A, α) condition $D_{i,i+1}$ holds and let (B, β) be isotopic to (A, α). Condition $D_{i,i+1}$ holds in (B, β) too. Let be e a unit in (B, β). From $\beta((e)^{i-1}, e, x_i, (e)^{n-i-1}) = \beta((e)^{i-1}, x_i, e, (e)^{n-i-1})$ we get $\beta(x_1^{i-1}, e, x_i, x_{i+1}^{n-1}) = \beta(x_1^{i-1}, x_i, e, x_{i+1}^{n-1})$.

Now suppose that in every n-loop (B, β) isotopic to (A, α) $\beta(x_1^{i-1}, e, x_i^{n-1}) = \beta(x_1^i, e, x_{i+1}^{n-1})$ holds for each unit e. We prove that in

(A, α) condition $D_{i,i+1}$ holds. Let be

$$\alpha(a_1^{i-1}, u_i, u_{i+1}, a_{i+2}^n) = \alpha(a_1^{i-1}, v_i, v_{i+1}, a_{i+2}^n) \tag{1}$$

Define $a^* = (a_1^*, \ldots, a_n^*)$ as follows: $a_j^* = a_j$ for $j \in \{1, \ldots, n\} - \{i, i+1\}$, $a_i^* = u_i$, $a_{i+1}^* = v_{i+1}$. Using translations T_i by a^* we define the LP-isotope (A, β) of (A, α). Equality (1) can be written

$$T_{i+1}(u_{i+1}) = T_i(v_i) \tag{2}$$

Note that $e = \alpha(a_1^*, \ldots, a_n^*) = T_i(u_i) = T_{i+1}(v_{i+1})$ is a unit in (A, β). Now

$$\begin{aligned} &\alpha(x_1^{i-1}, u_i, u_{i+1}, x_{i+1}^n) = \\ = \;& \beta(T_1(x_1), \ldots, T_{i-1}(x_{i-1}), e, T_{i+1}(u_{i+1}), T_{i+2}(x_{i+2}), \ldots, T_n(x_n)) = \\ = \;& \beta(T_1(x_1), \ldots, T_{i-1}(x_{i-1}), T_{i+1}(u_{i+1}), e, T_{i+2}(x_{i+2}), \ldots, T_n(x_n)) = \\ = \;& \beta(T_1(x_1), \ldots, T_{i-1}(x_{i-1}), T_i(v_i), T_{i+1}(v_{i+1}), T_{i+2}(x_{i+2}), \ldots, T_n(x_n)) = \\ = \;& \alpha(x_1^{i-1}, v_i, v_{i+1}, x_{i+2}^n). \end{aligned}$$

Based on Theorem 4 we prove the following criterion for an n-quasigroup to be a G-quasigroup.

Theorem 5. *(A, α) is a G-quasigroup if and only if condition $D_{1,2} \& D_{2,3} \& \ldots \& D_{n-1,n}$ holds.*

Proof. Let (A, β) be an n-loop isotopic to (A, α) and e a unit in (A, β). In (A, β) condition $D_{1,2} \& \ldots \& D_{n-1,n}$ holds too. By Theorem 4 e is in the center of (A, β). Define $x \cdot y = \beta(x, y, (e)^{n-2})$. It is obvious that $(A, \cdot)$ is a binary quasigroup. From $\beta(x_1, x_2, (e)^{n-2}) = \beta(\beta(x_1, x_2, (e)^{n-2}), e, (e)^{n-2})$ by $D_{1,2}$ we get

$$\beta(x_1^n) = \beta(x_1 x_2, e, x_3^n). \tag{3}$$

Analogously, from $\beta((e)^{n-2}, x_{n-1}, x_n) = \beta((e)^{n-2}, e, \beta((e)^{n-2}, x_{n-1}, x_n))$ by $D_{n-1,n}$ we have

$$\beta(x_1^n) = \beta(x_1^{n-2}, e, x_{n-1} x_n). \tag{4}$$

Taking into account (3),

$$\begin{aligned} \beta(x_1, x_2, x_3, (e)^{n-3}) &= \beta(x_1 x_2, e, x_3, (e)^{n-3}) = \beta(x_1 x_2, x_3, (e)^{n-2}) = \\ &= \beta((x_1 x_2) x_3, (e)^{n-1}) = (x_1 x_2) x_3. \end{aligned}$$

Analogously, using (4) we get $\beta((e)^{n-3}, x_1, x_2, x_3) = x_1(x_2x_3)$. Therefore $(x_1x_2)x_3 = x_1(x_2x_3)$ (e is in the center), i.e. $(A, \cdot)$ is a binary group.

Continuing the above procedure we obtain $\beta(x_1^n) = x_1x_2\ldots x_n$.

The converse statement is obvious. In any n-group derived from a binary group any condition $D_{i,j}$ holds.

From the above results we obtain the following characterization of G-3-quasigroups.

Theorem 6. *A 3-quasigroup (A, α) is a G-quasigroup iff there exist four binary quasigroups (A, α_i), such that*

$$\alpha_1(\alpha_2(x, y), z) = \alpha_3(x, \alpha_4(y, z)) = \alpha(x, y, z)$$

for all $x, y, z \in A$.

Proof. Suppose (A, α) be a G-quasigroup. By Theorem 5 in (A, α) condition $D_{1,2}\&D_{2,3}$ holds. By Theorem 3 condition $D_{1,2}$ implies $\alpha(x, y, z) = \alpha_1(\alpha_2(x, y), z)$ and condition $D_{2,3}$ implies $\alpha(x, y, z) = \alpha_3(x, \alpha_4(y, z))$.

The converse statement is clear.

Remark 1. The functional equation of generalized associativity on quasigroups: find the set of all solutions of the functional equation $\alpha_1(\alpha_2(x, y), z) = \alpha_3(x, \alpha_4(y, z))$, over the set of quasigroup operations on an arbitrary set A. Theorem 6 suggests a possibility to solve this equation using G-3-quasigroups.

2. $G_a - n$-quasigroups

In this section we present criteria for an n-quasigroup to be a G_a-quasigroup. We finish this section showing that G_a-4-quasigroups are connected with the functional equation of generalized bisymmetry.

Let (A, α) be an n-quasigroup.

Definition 4. *We say that in (A, α) condition D_{i-j}, $1 \leq i,\ j \leq n$, $i + 1 < j$, holds if $\alpha(a_1^{i-1}, u_i, a_{i+1}^{j-1}, u_j, a_{j+1}^n) = \alpha(a_1^{i-1}, v_i, a_{i+1}^{j-1}, v_j, a_{j+1}^n)$ implies $\alpha(x_1^{i-1}, u_i, x_{i+1}^{j-1}, u_j, x_{j+1}^n) = \alpha(x_1^{i-1}, v_i, x_{i+1}^{j-1}, v_j, x_{j+1}^n)$ for all $x_1^n \in A^n$.*

It is easy to prove that condition D_{i-j} is isotopic invariant.

Theorem 7. *In (A, α) condition D_{i-j} holds iff there exist two quasigroups, (A, γ) of arity $n - 1$ and a binary quasigroup (A, β) such that*

$$\alpha(x_1^n) = \gamma(x_1^{i-1}, \beta(x_i, x_j), x_{i+1}^{j-1}, x_{j+1}^n).$$

Proof. We arbitrary choose $a_j \in A$. For any $x_1^n \in A^n$ there exists exactly one $b \in A$ such that $\alpha(x_1^n) = \alpha(x_1^{i-1}, b, x_{i+1}^{j-1}, a_j, x_{j+1}^n)$. By condition D_{i-j} we get $\alpha(y_1^{i-1}, x_i, y_{i+1}^{j-1}, x_j, y_{j+1}^n) = \alpha(y_1^{i-1}, b, y_{i+1}^{j-1}, a_j, y_{j+1}^n)$. Hence b depends only of x_i and x_j. Putting $b = \beta(x_i, x_j)$ we obtain

$$\alpha(x_1^n) = \alpha(x_1^{i-1}, \beta(x_i, x_j), x_{i+1}^{j-1}, a_j, x_{j+1}^n) = \gamma(x_1^{i-1}, \beta(x_i, x_j), x_{i+1}^{j-1}, x_{j+1}^n)$$

where $\gamma(x_1^{n-1}) = \alpha(x_1^{j-1}, a_j, x_{j+1}^n)$ is a retract of α.

Theorem 8. *In (A, α) condition D_{i-j} holds iff for every n-loop (B, β) isotopic to (A, α), $\beta(x_1^{i-1}, e, x_i^{n-1}) = \beta(x_1^{i-1}, x_j, x_{i+1}^{j-1}, e, x_{j+1}^n)$ for each unit e and all $x_1^n \in B$.*

Proof. Similar to the proof of Theorem 4.

Theorem 9. *(A, α) is a G_a-quasigroup iff it is a G-quasigroup and a condition D_{i-j} holds.*

Proof. We prove that $(A, \cdot)$ (see the proof of Theorem 5) is commutative: $xy = \beta((e)^{i-1}, x, y, (e)^{n-i-1}) = \beta((e)^{i-1}, e, y, (e)^{j-i-2}, x, (e)^{n-j}) = yx$.

If $n > 3$ we can replace a condition $D_{i,i+1}$, $1 < i < n-1$ by conditions $D_{(i-1)-(i+1)}$ and $D_{i-(i+2)}$.

Theorem 10. *An n-quasigroup (A, α), $n > 3$ is a G_a-quasigroup iff condition $D_{1,2} \& \ldots \& D_{(i-1),i} \& D_{(i-1)-(i+1)} \& D_{i-(i+2)} \& D_{i+1,i+2} \& \ldots \& D_{n-1,n}$ holds.*

Proof. The proof is analogous to the proof of Theorem 5.

We finish by a characterization of G_a-4-quasigroups.

Theorem 11. *A 4-quasigroup (A, α) is a G_a-quasigroup iff there exist six binary quasigroups (A, α_i) such that*

$$\alpha_1(\alpha_2(x, y), \alpha_3(u, v)) = \alpha_4(\alpha_5(x, u), \alpha_6(y, v)) = \alpha(x, y, u, v).$$

Proof. By Theorem 10 (A, α) is a G_a-quasigroup iff $D_{12} \& D_{1-3} \& D_{2-4} \& D_{34}$ holds.

By Theorem 3 condition $D_{1,2}$ implies $\alpha(x, y, u, v) = \beta(\alpha_2(x, y), u, v)$ where $\beta(x, y, z) = \alpha(a, x, y, z)$, $a \in A$. It is easy to prove that if in (A, α) condition $D_{3,4}$ holds then in (A, β) condition $D_{2,3}$ holds too. Again by Theorem 3 we obtain $\beta(x, y, z) = \alpha_1(x, \alpha_3(y, z))$ and then $\alpha(x, y, u, v) = \alpha_1(\alpha_2(x, y), \alpha_3(u, v))$.

Now by Theorem 7 condition D_{1-3} implies $\alpha(x, y, u, v) = \gamma(\alpha_5(x, u), y, v)$ where $\gamma(x, y, z) = \alpha(x, y, a, z)$, $a \in A$. It is not difficult to prove if in

(A, α) condition D_{2-4} holds then condition $D_{2,3}$ holds in (A, γ). Applying Theorem 3 we have $\gamma(x, y, z) = \alpha_4(x, \alpha_6(y, z))$. Hence $\alpha(x, y, u, v) = \alpha_4(\alpha_5(x, u), \alpha_6(y, v))$.

The converse is obvious.

Remark 2. The functional equation of generalized bisymmetry on quasigroups: find the set of all solutions of the functional equation $\alpha_1(\alpha_2(x, y), \alpha_3(u, v)) = \alpha_4(\alpha_5(x, u), \alpha_6(y, v))$ over the set of quasigroup operations on an arbitrary set A.

Theorem 11 suggests a possibility to solve this equation using G_a-4-quasigroups.

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