ON THE FROBENIUS NUMBER OF SOME LUCAS NUMERICAL SEMIGROUPS

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Abstract. In this study, the results on the Lucas numbers are given, and the Frobenius number of some Lucas numerical semigroups is computed.

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Introduction

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the set of integers and nonnegative integers, respectively. A numerical semigroup S is a subset of $\mathbb{N}$ that is closed under addition, $0 \in S$, and generates $\mathbb{Z}$ as a group. There exist elements of S , say $s_0, s_1, \ldots, s_p$ such that $s_0 < s_1 < \ldots < s_p$ and

$$S =< s_0, s_1, ..., s_p >= \{k_0 s_0 + k_1 s_1 + ... + k_p s_p : \ k_i \in \mathbb{N}, s_i \in S, 0 \leq i \leq p\}.$$

From this definition, we obtain that the set $\mathbb{N} \backslash S$ is finite, and we say that $\{ s_0 , s_1, \ldots, s_p \}$ is a the minimal system of generator for S . The *Frobenius number* of S, denoted by $g(S)$, is the largest integer not in S. That is, $g (S) = \max \{ x \in \mathbb{Z} : \ x \notin S \}$ (see [1]).

Thus, a numerical semigroup S can be expressed as

$$S = \{ 0, s_0, s_1, \ldots, s_p, \ldots, g(S) + 1, \rightarrow \ldots \}$$

where "$\rightarrow$" means that every integer greater than $g(S)+1$ belongs to S .We say that a numerical semigroup is *symmetric* if we have $(g (S) - x) \in S$, for every $x \in \mathbb{Z} \backslash S$.

If numerical semigroup S is generated by a and b elements then, we know that S is symmetric and $g(S) = ab - a - b$ (See [2]).

The number of Fibonacci n is denoted as $F(n) = (n-1) + (n-2)$ for $n > 1$, where $F(0) = 0$ and $F(1) = 1$. Also denoted as $F(n)$, for $n = 0, 1, ...$ are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, \ldots$$

We say that a number n is Lucas if $L(n) = F(n-1) + F(n+1)$ for $n > 1$, where $L(0) = 2$ and $L(1) = 1$. Also denoted as $L(n)$, for $n = 0, 1, ...$ are

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, ...$$

This study consists of two sections. In section I and II, we give some properties of Lucas numbers and some criteria to calculate the Frobenius number of the Lucas numerical semigroups, respectively.

1. Some results for Lucas Numbers

We will use F_n and L_n instead of $F(n)$ and $L(n)$, respectively.

Lucas numbers are related to Fibonacci numbers by relation $L_n = F_{n-1} + 3F_n$. We can obtain the similar relations between Lucas and Fibonacci numbers. We known that the following equalities for Fibonacci numbers are true (See [3]);

1) $F_0 + F_1 + F_2 + ... + F_n = F_{n+2} - 1$
2) $F_1 + 2F_2 + 3F_3 + ... + nF_n = n\,F_{n+2} - F_{n+3} + 2$
3) F_{3n} is even, for $n \geq 1$.

Now, we give the similar properties for Lucas numbers.

Theorem 1.1.

$$L_1 + L_2 + \ldots + L_n = L_{n+2} - 3 \tag{1.1}$$

Proof. Let's prove the theorem by the induction on n .

The equality (1.1) is true for $n = 1$ since $L_1 = L_3 - 3 = 4 - 3 = 1$.

We assume that (1.1) is true for $n = k$. Then, we must show that it is also true for $n = k+1$. For $n = k$, we can write that $L_1 + L_2 + \ldots + L_k = L_{k+2} - 3$. With L_{k+1} to be added on both ides of this equality, we obtain

$$\begin{aligned} L_1 + L_2 + \ldots + L_k + L_{k+1} &= L_{k+2} - 3 + L_{k+1} \\ &= 2F_{k+3} - F_{k+2} + 2F_{k+2} - F_{k+1} - 3 \\ &= 2F_{k+3} + F_{k+2} - F_{k+1} - 3 \\ &= 2(F_{k+2} + F_{k+1}) + F_{k+2} - F_{k+1} - 3 \\ &= 3F_{k+2} + F_{k+1} - 3 \\ &= 2F_{k+2} - 3 + (F_{k+2} + F_{k+1}) \\ &= 2F_{k+2} + F_{k+3} - 3 \\ &= (F_{k+2} + F_{k+3}) + F_{k+2} - 3 \\ &= F_{k+4} + F_{k+2} - 3\,. \end{aligned}$$

Theorem 1.2.

$$L_1 + 2L_2 + 3L_3 + \ldots + n\,L_n = nL_{n+2} - L_{n+3} + 4 \tag{1.2}$$

Proof. We make proof of theorem same above operations.

Theorem 1.3. *L_{3n} is even, for $n \geq 1$.*

Proof. We can write $L_{3n} = 2L_{3n-2} + L_{3n-3}$. For $n = 1$, $L_3 = L_2 + L_1 = 3 + 1 = 4$ is even. Now, we suppose that $L_{3k} = 2L_{3k-2} + L_{3k-3}$ is even for $n = k$. Putting $n = k + 1$, then we obtain that $L_{3(k+1)} = L_{3k+22} + L_{3k+13} = 2L_{3k+1} + L_{3k}$. Therefore, $L_{L3(k+1)}$ is even.

2. The Frobenius number of Lucas numerical semigroups

In this section, we will give some results on the Frobenius numbers of certain Lucas numerical semigroups which are generated by Lucas numbers.

Theorem 2.1. *Let $S =< L_n\,, L_{n+1}\,, L_{n+k} >$ for $n, k \geq 2$. Then, the Frobenius number of Lucas numerical semigroup S is $g(S) = L_nL_{n+1} - L_n - L_{n+1}$. However, S is symmetric.*

Proof. Using properties of Lucas numbers , it is not difficult to show that there exist $a\,, b \in IN$ such that $L_{n+k} = a.L_n + b.L_{n+1}$ for $n\,, k \geq 2$. For example, let $k = 2$. Then, we find that

$$\begin{aligned} L_{n+2} &= F_{n+1} + 3F_{n+2} = (3F_{n41} + F_n) + F_{n+1} + 2F_n \\ &= (3F_{n41} + F_n) + (3F_n + F_{n-1}) \\ &= L_{n+1} + L_n\,. \end{aligned}$$

Thus, we can write that $S =< L_n, L_{n+1}, L_{n+k} >=< L_n, L_{n+1} >$. Therefore, we can get the minimal system of generator of Lucas numerical semigroup $S =< L_n\ , L_{n+1}, L_{n+k} >$ is $\{L_n\ ,\ L_{n+1}\}$. In this case, the Frobenius number of Lucas numerical semigroup S is $g(S) = L_nL_{n+1} - L_n - L_{n+1}$. Also, S is symmetric since $S =< L_n, L_{n+1}, L_{n+k} >=< L_n, L_{n+1} >$.

Theorem 2.2. *Let $S =< L_n\ , L_{n+2}\ , L_{n+3}\ >$ for $n \geq 3$. Then, the Frobenius number of Lucas numerical semigroup S is $g(S\ ) = L_n.(\ \lfloor \frac{L_n-2}{2} \rfloor\ ) + L_{n+1}(\ L_n - 1\ )$, where $\lfloor x \rfloor$ is the greatest integer and smaller than x , for x rational number.*

Proof. Using the properties of Lucas numbers, we have

$$S =< L_n, L_{n+2}, L_{n+3} >=< L_n, L_n + L_{n+1}, L_n + 2L_{n+1} >$$

for $n \geq 3$. If we put $L_n = a$ and $L_{n+1} = d$, we can write $S =< a\ , a + d,\ a + 2d\ >$. Hence, we find the Frobenius number of S Lucas numerical semigroups S as

$$g(S) = a(\left\lfloor \frac{a-2}{2} \right\rfloor) + d\,(a-1)$$

(see [4]).

Example 2.3. Let $S =< L_4\ , L_6\ , L_7\ >=< 7\ , 18\ , 29\ >= \{0, 7\ , 14, 18, 21, 25, 28, 29, 32, 35, 36, 39, 42, 43, 46, 47, 49, 50, 53, 54, 56, 57, 58, 60, 61, 63, 64, 65, 67, 68, 70, 71, 72, 74, 75, 76, 77, 78, 79, 81, \rightarrow ...\}$ for $n = 4$. Then, the Frobenius number of Lucas numerical semigroup S is

$$g(S) = L_4(\left\lfloor \frac{L_4-2}{2} \right\rfloor) + L_5(L_4-1) = 7(\left\lfloor \frac{7-2}{2} \right\rfloor) + 11(7-1) = 7.2 + 11.6 = 80.$$

However, S is not symmetric since $g(S) - 11 = 80 - 11 = 69 \notin S$ for $11 \notin S$.

Theorem 2.4. *Let $S =< L_{3n}\ , L_{3n}+2\ , 2L_{3n}+1\ >$ for $n \geq 1$. Then, the Frobenius number of Lucas numerical semigroup S is $g(S\ ) = \frac{(L_{3n})^2}{2} + L_{3n} - 1$. Also, S is symmetric.*

These numerical semigroups are known as telescopic and they are symmetric (see [5]).

Proof. Let $L_{3n} = a$. Then, we can write $S = < a, a+2, 2a+1 >$. By Theorem 1.3, we find gcd $\{a, a+2\} = 2$ since a is even. Thus, we can write

$$2a + 1 = 3(\frac{a}{2}) + 1(\frac{a+2}{2}) \in < \frac{a}{2}, \frac{a+2}{2} > .$$

In this case, the Frobenius number of Lucas numerical semigroup S is $g(S) = \frac{a^2}{2} + a - 1 = \frac{(L_{3n})^2}{2} + L_{3n} - 1$. Also, S is symmetric.

Example 2.5. Let $S = < L_6, L_6 + 2, 2L_6 + 1 > = < 18, 20, 37 >$. Then S Lucas numerical semigroup is telescopic since $37 \in < 9, 10 >$. Thus, the Frobenius number of Lucas numerical semigroup S is $g(S) = \frac{18^2}{2} + 18 - 1 = 179$. Also, S is symmetric since $179 - x \in S$ for $\forall x \in \mathbb{Z} \backslash S$.

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