A STUDY OF THE STATIONARY REACTIVE FLOW OF A FLUID COFINED IN N-DIMENSIONAL DOMAINS WITH HOLES USING FIXED POINT THEORY

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Abstract. Motivated by a lot of type of chemical reactions which take place in domains with holes, a mathematical model is constructed, then the unique solvability is proved, using fixed point arguments.

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1. Introduction

We study here the existence and the uniqueness of the solution of a transmission problem in some chemical reactive flows through perforated domains.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain. Then a set of periodically holes in Ω with boundary S^ε are considered and denote

$$\Omega^\varepsilon = \Omega - \cup S^\varepsilon \quad , \quad \Pi^\varepsilon = \Omega - \overline{\Omega^\varepsilon}.$$

The holes S^ε are of size ε, where $\varepsilon > 0$ is a small parameter. In practical case, the holes are fulfilled with a granular material and the reactive fluid can penetrate inside the grains, where chemical reactions take place. If denote by u^ε the concentration of the reactive fluid cofined in Ω^ε and by v^ε the concentration inside the grains, then the chemical reactions are governed by the following relations:

$$\begin{cases} -D_f \Delta u^\varepsilon = f(u^\varepsilon) & , \quad \text{in} \quad \Omega^\varepsilon \\ -D_f \Delta v^\varepsilon + a g(v^\varepsilon) = 0 & , \quad \text{in} \quad \Pi^\varepsilon \\ -D_f \cdot \dfrac{\partial u^\varepsilon}{\partial v} = D_p \cdot \dfrac{\partial v^\varepsilon}{\partial v} & , \quad \text{on} \quad S^\varepsilon \\ u^\varepsilon = v^\varepsilon & , \quad \text{on} \quad S^\varepsilon \\ u^\varepsilon = 0 & , \quad \text{on} \quad \partial\Omega \end{cases} \tag{1.1}$$

where v is the exterior normal to Ω^ε, while $a > 0$ and D_f, D_p are some constant diffusion coefficients, characterizing the reactive fluid, respective the granular material from inside the holes. As in models of Langmuir kinetics [3] or in Freundlich kinetics [2], where

$$g(v) = \frac{\alpha v}{1 + \beta v} \quad (\alpha, \beta > 0) \quad , \quad \text{respective } g(v) = |v|^{p-1} \cdot v \quad (0 < p < 1),$$

the function g is in generally assumed to be continuous, monotone increasing, while f is monotone increasing and continuously-differentiable.

In this model (1.1), the function $\begin{pmatrix} u^\varepsilon \\ v^\varepsilon \end{pmatrix}$, defined on

$$u^\varepsilon : \Omega^\varepsilon \to \mathbb{R} \quad , \quad v^\varepsilon : \Pi^\varepsilon \to \mathbb{R}$$

converges weakly in the Sobolev space $H_0^1(\Omega)$ to the solution of the following elliptic problem:

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{n} a_{ij} \cdot \frac{\partial^2 u}{\partial x_i \partial x_j} + q g(u) = f(u) & , \quad \text{in} \quad \Omega \\ \qquad\qquad u = 0 & , \quad \text{on} \quad \partial\Omega \end{cases} , \tag{1.2}$$

where $(a_{ij})_{1 \le i,j \le n}$ is the homogenized, positive defined matrix and $q > 0$.

2. The result

In order to study the problem (1.2), we consider $\alpha > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \ge \alpha |\xi|^2 \quad , \quad \text{for every } \xi \in \mathbb{R}^n$$

and we will define the following strongly elliptic problem

$$\begin{cases} -\sum_{i,j=1}^{n} a_{ij} \cdot \frac{\partial^2 u}{\partial x_i \partial x_j} + g(x,u) = f(x) & in \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases} \tag{2.1}$$

which is more generally than the problem (1.2). Mention that by considering $g(x,u)$ depending also on x, we solve other more complicated diffusion problems arising in chemistry or physics. The main result of this work is the following

Theorem 2.1. *If* $f \in L^2(\Omega)$ *and* $g(x,u)$ *has partial derivative in* u *of the first order with*

$$m \le \frac{\partial g}{\partial u} \le M \quad , \quad in \ \ \Omega, \tag{2.2}$$

for some $m, M > 0$, *then the problem (2.1) has an unique weak solution.*

Let us define the operator $A : D(A) \subset H \to H$ by the formula

$$Au = -\sum_{i,j=1}^{n} a_{ij} \cdot \frac{\partial^2 u}{\partial x_i \partial x_j},$$

where

$$H = L^2(\Omega) \quad , \quad D(A) := H^2(\Omega) \cap H_0^1(\Omega)$$

and denote $F(u) := g(\cdot, u) - f$. The operator A is monotone:

$$(Au, u) = \sum_{i,j=1}^{n} \int_{\Omega} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} \ge 0$$

and $I + A$ is surjective ([1], p.177), thus the operator A is maximal monotone. From the relation (2.2) it follows

$$< F(u) - F(v), u - v > \ge m \cdot |u - v|^2 \tag{2.3}$$

and

$$|F(u) - F(v)| \le M \cdot |u - v| \, , \tag{2.4}$$

using a Lagrange type theorem. Now the problem (2.1) can be written in the following abstract form:

$$Au + F(u) = 0 \quad , \quad \text{in} \ \ L^2(\Omega), \ \ \text{with } u \in H^2(\Omega) \cap H_0^1(\Omega) \tag{2.5}$$

Proof of the Theorem 2.1 Let us consider the problem (2.1) as a semilinear equation of the form (2.5). We show first that there exists $\lambda > 0$ such that

$$S_\lambda : H \to H \quad , \quad \text{given by} \;\; S_\lambda(u) := u - \lambda F(u)$$

is a contraction. In this sense, using the relations (2.3)-(2.4), we deduce that

$$\begin{aligned} & |S_\lambda(u) - S_\lambda(v)|^2 \\ & = |u - v|^2 - 2\lambda \cdot < F(u) - F(v), u - v > +\lambda^2 |F(u) - F(v)|^2 \\ & \leq (1 - 2\lambda m + \lambda^2 M) |u - v|^2 , \end{aligned}$$

thus

$$|S_\lambda(u) - S_\lambda(v)| \leq c \cdot |u - v| ,$$

with

$$c := \sqrt{1 - 2\lambda m + \lambda^2 M} < 1 \quad , \quad \text{if} \;\; \lambda \in (0, 2m/M).$$

Now the equation (2.5) can be written as

$$(I + \lambda A)u = S_\lambda(u), \tag{2.6}$$

where $\lambda > 0$ is so that S_λ is a contraction. Using the fact that $(I + \lambda A)$ is inversable and $|(I + \lambda A)^{-1}| \leq 1$ for each $\lambda > 0$ (because A is maximal monotone, e.g.[1], p.101) the equation (6) is equivalent with

$$u = (I + \lambda A)^{-1} S_\lambda(u).$$

We have

$$\begin{aligned} & \left|(I + \lambda A)^{-1} S_\lambda(u) - (I + \lambda A)^{-1} S_\lambda(v)\right| \\ & = \left|(I + \lambda A)^{-1}(S_\lambda(u) - S_\lambda(v))\right| \\ & \leq \left|(I + \lambda A)^{-1}\right| \cdot |S_\lambda(u) - S_\lambda(v)| \leq c \cdot |u - v| . \end{aligned}$$

Therefore, $u \mapsto (I + \lambda A)^{-1} S_\lambda(u)$ is a contraction having an unique fixed point, thus (2.5) and consequently (2.1) has an unique weak solution.

References

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