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PROPERTIES FOR A CLASS RELATED TO Q-FRACTIONAL DIFFERENTIAL OPERATOR

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Abstract. The object of this paper is to introduce and study modified Hadamard product and partial sums and some properties of a class defined by q-Al-Oboudi - Al-Amoudi operator.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in U=$\{z \in \mathbb{C} : |z| < 1\}$ and $\Im \subset \mathcal{A}$ for which

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0). \tag{1.2}$$

The $q-$difference operator for function $f(z)$ defined by $([1], [6,7], [9], [15] \text{ and } [17])$;

$$D_q = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & , z \neq 0 \\ f'(0) & , z = 0 \end{cases}, \tag{1.3}$$

that is

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^k, \tag{1.4}$$

where

$$[k]_q = \frac{1-q^k}{1-q}, [0]_q = 0. \tag{1.5}$$

Note that:

As $q \to 1^-$, $[k]_q = k$, $D_q f(z) = f'(z)$.

The fractional $q-$derivative operator of order α for analytic function f defined in a simply connected domain, contains zero is defined by [6],

$$D^{\alpha}_{q,z} f(z) = \frac{1}{\Gamma_q(1-\alpha)} \int_0^z \frac{f(t)}{(z-t)^{\alpha}} d_q t \ , \ 0 \le \alpha < 1,$$

$$\Omega^{\alpha}_q f(z) = \Gamma_q(2-\alpha) z^{\alpha} D^{\alpha}_{q,z} f(z),$$

$$= z + \sum_{k=2}^{\infty} \frac{\Gamma_q(k+1)\Gamma_q(2-\alpha)}{\Gamma_q(k+1-\alpha)} a_k z^k \ \ (0 < q < 1, \ 0 \le \alpha < 1),$$

where multiplicity of $(z-t)^{-\alpha}$ is removed by requiring $\log(z-t)$, to be real when $z - t$ ¿0 (for q $\to 1^-$ see [18], [19]).

For $\lambda \ge 0$, $0 \le \alpha < 1$, $0 < q < 1$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and f is given by (1.1) we defined new $q-$fractional derivative operator as follows,

$$\begin{aligned}
D^{0,0}_{\lambda,q} f(z) &= f(z), \\
D^{1,\alpha}_{\lambda,q} f(z) &= (1-\lambda)\Omega^{\alpha}_q f(z) + \lambda z D_q \left(\Omega^{\alpha}_q f(z)\right) = D^{\alpha}_{\lambda,q} f(z), \\
D^{2,\alpha}_{\lambda,q} f(z) &= D^{\alpha}_{\lambda,q}\left(D^{\alpha}_{\lambda,q} f(z)\right), \\
D^{n,\alpha}_{\lambda,q} f(z) &= D^{\alpha}_{\lambda,q}\left(D^{n-1,\alpha}_{\lambda,q} f(z)\right),
\end{aligned}$$

$$= z + \sum_{k=2}^{\infty} \Psi_{k,n,q}(\alpha,\lambda) a_k z^k, \tag{1.6}$$

where

$$\Psi_{k,n,q}(\alpha,\lambda) = \frac{\Gamma_q(k+1)\Gamma_q(2-\alpha)}{\Gamma_q(k+1-\alpha)} \left[1 + \lambda([k]_q - 1)\right]^n. \tag{1.7}$$

We note that for $q \to 1^-$

(i) $D^{n,\alpha}_{\lambda,q} = D^{n,\alpha}_{\lambda}$, where this operator modified the operator of [3, 8],

(ii) $D^{0,\alpha}_{\lambda} = D^{\alpha}_{z}$ (see [18], [19]),

(*iii*) $D_1^{n,0} = D^n$ [20],
(*iv*) $D_\lambda^{n,0} = D_\lambda^n$ [2].

Definition 1. *For $\lambda, \mu \geq 0$, $\gamma \geq 1$, $0 \leq \alpha, \beta < 1$, $0 \leq \delta \leq 1$, $n \in \mathbb{N}_0$ and a function $f \in \mathcal{A}$ is in the class $S_{\lambda,q}^{n,\alpha}(\delta, \gamma, \mu, \beta)$, if it satisfies*

$$Re\left\{\frac{\gamma z D_q G(z)}{G(z)} - (\gamma - 1)\right\} > \mu \left|\frac{\gamma z D_q G(z)}{G(z)} - \gamma\right| + \beta, \tag{1.8}$$

where

$$G(z) = (1-\delta) D_{\lambda,q}^{n,\alpha} f(z) + \delta z \left(D_q D_{\lambda,q}^{n,\alpha} f(z)\right). \tag{1.9}$$

Let

$$TS_{\lambda,q}^{n,\alpha}(\delta, \gamma, \mu, \beta) := S_{\lambda,q}^{n,\alpha}(\delta, \gamma, \mu, \beta) \cap \Im. \tag{1.10}$$

We note that as $q \to 1^-$:
$S_{\lambda,q}^{n,\alpha}(0, 1, \mu, \beta) = SP_{\alpha,\lambda}^n(\mu, \beta)$ and $S_{\lambda,q}^{n,\alpha}(1, 1, \mu, \beta) = UCV_{\alpha,\lambda}^n(\mu, \beta)$ [3, 8, with $\Psi_{k,n,q}(\alpha, \lambda)$ of the form (1.7)]. For different values of $n, \alpha, \lambda, \delta, \gamma, \mu$ and β, we get the classes defined by [8] and [10 − 13].

2. Coefficient estimate

In the rest of the paper let $0 \leq \alpha, \beta < 1$, $\lambda, \mu \geq 0$, $\gamma \geq 1, 0 \leq \delta \leq 1$, $n \in \mathbb{N}_0$, $0 < q < 1$ and $\Psi_{k,n,q}(\alpha, \lambda)$ as (1.7).

Theorem 1. *If $f \in \mathcal{A}$, satisfies*

$$\sum_{k=2}^{\infty} \left[1 - \beta + \gamma\left([k]_q - 1\right)(1+\mu)\right]\left[1 + \left([k]_q - 1\right)\delta\right] \Psi_{k,n,q}(\alpha, \lambda) |a_k| \leq 1 - \beta, \tag{2.1}$$

then $f \in S_{\lambda,q}^{n,\alpha}(\delta, \gamma, \mu, \beta)$.

Proof. Assume that (2.1) hold. Since for real β and complex number w,

$$R(w) \geq \beta \Leftrightarrow |w + (1-\beta)| - |w - (1+\beta)| \ \geq 0, \tag{2.2}$$

then by Definition 1 it is sufficient to show that

$$\left| \frac{\gamma z D_q G(z)}{G(z)} - (\gamma - 1) - \mu \left| \frac{\gamma z D_q G(z)}{G(z)} - \gamma \right| - (1+\beta) \right| \leq$$
$$\left| \frac{\gamma z D_q G(z)}{G(z)} - (\gamma - 1) - \mu \left| \frac{\gamma z D_q G(z)}{G(z)} - \gamma \right| + (1-\beta) \right|. \tag{2.3}$$

For the right-hand side of (2.3)

$$\begin{aligned} R \quad : \quad &= \left| \frac{\gamma z D_q G(z)}{G(z)} - (\gamma - 1) - \mu \left| \frac{\gamma z D_q G(z)}{G(z)} - \gamma \right| + (1-\beta) \right| \\ &= \frac{1}{|G(z)|} \left| \gamma z D_q G(z) + (2 - \beta - \gamma) G(z) - \mu e^{i\theta} \left| \gamma z D_q G(z) - \gamma G(z) \right| \right| \\ &> \frac{|z|}{|G(z)|} \{ 2 - \beta - \sum_{k=2}^{\infty} \left[2 - \beta + \gamma \left([k]_q - 1 \right) (1+\mu) \right] \left[1 + \left([k]_q - 1 \right) \delta \right] \Psi_{k,n,q}(\alpha, \lambda) \, |a_k| \}. \end{aligned}$$

Similarly, the left

$$\begin{aligned} L \quad : \quad &= \left| \frac{\gamma z D_q G(z)}{G(z)} - (\gamma - 1) - \mu \left| \frac{\gamma z D_q G(z)}{G(z)} - \gamma \right| - (1+\beta) \right| \\ &= \frac{1}{|G(z)|} \left| \gamma z D_q G(z) - (\gamma - 1) G(z) - \mu e^{i\theta} \left| \gamma z D_q G(z) - \gamma G(z) \right| - (1+\beta) G(z) \right| \\ &< \frac{|z|}{|G(z)|} \{ \beta + \sum_{k=2}^{\infty} \left[\gamma \left([k]_q - 1 \right) (1+\mu) - \beta \right] \left[1 + \left([k]_q - 1 \right) \delta \right] \Psi_{k,n,q}(\alpha, \lambda) \, |a_k| \}. \end{aligned}$$

Since

$$R - L >$$

$$\frac{|z|}{|G(z)|} \{ 2(1-\beta) - 2 \sum_{k=2}^{\infty} \left[1 - \beta + \gamma \left([k]_q - 1 \right) (1+\mu) \right] \left[1 + \left([k]_q - 1 \right) \delta \right] \Psi_{k,n,q}(\alpha, \lambda) \, |a_k| \} \geq 0,$$

then the required condition (2.3) is satisfied, so $f \in S_{\lambda,q}^{n,\alpha}(\delta, \gamma, \mu, \beta)$.

Theorem 2. *Let* $f \in \Im$, *then* $f \in TS_{\lambda,q}^{n,\alpha}(\delta,\gamma,\mu,\beta)$, *if and only if*

$$\sum_{k=2}^{\infty}\left[1-\beta+\gamma\left([k]_q-1\right)(1+\mu)\right]\left[1+\left([k]_q-1\right)\delta\right]\Psi_{k,n,q}(\alpha,\lambda)a_k \leq 1-\beta \,. \tag{2.4}$$

Proof. Assume that (2.4) holds and in virtue of Theorem 2.1, then $f \in TS_{\lambda,q}^{n,\alpha}(\delta,\gamma,\mu,\beta)$.

Conversely, suppose $f \in TS_{\lambda,q}^{n,\alpha}(\delta,\gamma,\mu,\beta)$, choosing the values of z, on the positive real axis the inequality (1.8) reduces to:

$$\frac{1-\sum_{k=2}^{\infty}\left[1+\left([k]_q-1\right)\gamma\right]\left[1+\left([k]_q-1\right)\delta\right]\Psi_{k,n,q}(\alpha,\lambda)a_k z^{k-1}}{1-\sum_{k=2}^{\infty}\left[1+\left([k]_q-1\right)\delta\right]\Psi_{k,n,q}(\alpha,\lambda)a_k z^{k-1}}-\beta>$$

$$\mu\left|\frac{\sum_{k=2}^{\infty}\left[1+\left([k]_q-1\right)\gamma\right]\left[1+\left([k]_q-1\right)\delta\right]\Psi_{k,n,q}(\alpha,\lambda)a_k z^{k-1}}{1-\sum_{k=2}^{\infty}\left[1+\left([k]_q-1\right)\delta\right]\Psi_{k,n,q}(\alpha,\lambda)a_k z^{k-1}}\right|.$$

Letting z$\longrightarrow 1^{-}$, we obtain the desired inequality.

3. Modified Hadamard products for the class $TS_{\lambda,q}^{n,\alpha}(\delta,\gamma,\mu,\beta)$

Let $f_j\,(j=1,2,...,p)$ be defined by

$$f_j(z)=z-\sum_{k=2}^{\infty}a_{k,j}z^k \quad (a_{k,j}\geq 0)\,. \tag{3.1}$$

The modified Hadamard product of f_1 and f_2 is defined by

$$(f_1 * f_2)(z)=z-\sum_{k=2}^{\infty}a_{k,1}a_{k,2}z^k\,. \tag{3.2}$$

Theorem 3. *Let* $f_j(z) \in TS_{\lambda,q}^{n,\alpha}(\delta,\gamma,\mu,\beta)$, $(j=1,2)$ *defined by* (3.1), *then* $(f_1 * f_2)(z) \in TS_{\lambda,q}^{n,\alpha}(\delta,\gamma,\mu,\rho)$, *where*

$$\rho = 1 - \frac{\gamma q\,(1+\mu)\,(1-\beta)^2}{[1-\beta+\gamma q\,(1+\mu)]^2\,(1+\delta q)\,\Psi_{2,n,q}\,(\alpha,\lambda) - (1-\beta)^2}. \tag{3.3}$$

The result is sharp.

Proof. Employing the techniques used by Schild and Silverman [21] , we need to find the largest ρ such that.

$$\sum_{k=2}^{\infty} \frac{\left[1-\rho+\gamma\left([k]_q-1\right)(1+\mu)\right]\left[1+\left([k]_q-1\right)\delta\right]\Psi_{k,n,q}(\alpha,\lambda)}{1-\rho} a_{k,1}a_{k,2} \leq 1. \tag{3.4}$$

Since

$$\sum_{k=2}^{\infty} \frac{\left[1-\beta+\gamma\left([k]_q-1\right)(1+\mu)\right]\left[1+\left([k]_q-1\right)\delta\right]\Psi_{k,n,q}(\alpha,\lambda)}{1-\beta} a_{k,j} \leq 1 \ \ (j=1,2)\,. \tag{3.5}$$

Then Cauchy-Schwarz inequality yields

$$\sum_{k=2}^{\infty} \frac{\left[1-\beta+\gamma\left([k]_q-1\right)(1+\mu)\right]\left[1+\left([k]_q-1\right)\delta\right]\Psi_{k,n,q}(\alpha,\lambda)}{1-\beta} \sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{3.6}$$

Thus it sufficies to show that

$$\sum_{k=2}^{\infty} \frac{\left[1-\rho+\gamma\left([k]_q-1\right)(1+\mu)\right]\left[1+\left([k]_q-1\right)\delta\right]\Psi_{k,n,q}(\alpha,\lambda)}{1-\rho} a_{k,1}a_{k,2} \leq$$

$$\sum_{k=2}^{\infty} \frac{\left[1-\beta+\gamma\left([k]_q-1\right)(1+\mu)\right]\left[1+\left([k]_q-1\right)\delta\right]\Psi_{k,n,q}(\alpha,\lambda)}{1-\beta} \sqrt{a_{k,1}a_{k,2}}, \tag{3.7}$$

that is

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(1-\rho)\left[1-\beta+\gamma\left([k]_q-1\right)(1+\mu)\right]}{(1-\beta)\left[1-\rho+\gamma\left([k]_q-1\right)(1+\mu)\right]} \ \ (k\geq 2)\,. \tag{3.8}$$

From (3.6) and (3.8) , we need to prove that

$$\frac{1-\beta}{\left[1-\beta+\gamma\left([k]_q-1\right)(1+\mu)\right]\left[1+\left([k]_q-1\right)\delta\right]\Psi_{k,n,q}(\alpha,\lambda)} \leq \frac{(1-\rho)\left[1-\beta+\gamma\left([k]_q-1\right)(1+\mu)\right]}{(1-\beta)\left[1-\rho+\gamma\left([k]_q-1\right)(1+\mu)\right]},$$

which leads to

$$\rho \leq 1-\frac{(1-\beta)^2\gamma\left([k]_q-1\right)(1+\mu)}{\left[1-\beta+\gamma\left([k]_q-1\right)(1+\mu)\right]^2\left[1+\left([k]_q-1\right)\delta\right]\Psi_{k,n,q}(\alpha,\lambda)-(1-\beta)^2}. \tag{3.9}$$

Since

$$\Phi_q(k)=1-\frac{(1-\beta)^2\gamma\left([k]_q-1\right)(1+\mu)}{\left[1-\beta+\gamma\left([k]_q-1\right)(1+\mu)\right]^2\left[1+\left([k]_q-1\right)\delta\right]\Psi_{k,n,q}(\alpha,\lambda)-(1-\beta)^2}, \tag{3.10}$$

is an increasing function of $(k \geq 2)$, letting $k=2$ in (3.10), we obtain

$$\rho \leq \Phi_q(2)=1-\frac{\gamma q(1+\mu)(1-\beta)^2}{[1-\beta+\gamma q(1+\mu)]^2(1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)-(1-\beta)^2}, \tag{3.11}$$

which proves the main assertion of Theorem 3.1. Finally,

$$f_j(z)=z-\frac{(1-\beta)}{[1-\beta+\gamma q(1+\mu)][1+\delta q]\Psi_{2,n,q}(\alpha,\lambda)}z^2 \quad (j=1,2), \tag{3.12}$$

give the sharpness.

Theorem 4. *If* $f_j(z) \in TS_{\lambda,q}^{n,\alpha}(\delta,\gamma,\mu,\beta_j)$, $(j=1,2)$ *defined by* (3.1), *then* $(f_1 * f_2)(z) \in TS_{\lambda,q}^{n,\alpha}(\delta,\gamma,\mu,\rho_1)$, *where*

$$\rho_1=1-\frac{\gamma q(1+\mu)(1-\beta_1)(1-\beta_2)}{[1-\beta_1+\gamma q(1+\mu)][1-\beta_2+\gamma q(1+\mu)](1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)-(1-\beta_1)(1-\beta_2)}. \tag{3.13}$$

The result is the best possible for $f_j(z)$,

$$f_j(z)=z-\frac{(1-\beta_j)}{[1-\beta_j+\gamma q(1+\mu)][1+q\delta]\Psi_{2,n,q}(\alpha,\lambda)}z^2 \quad (j=1,2). \tag{3.14}$$

Proof. Proceeding as in the proof of Theorem 3.1, we get

$$\rho_1 \leq \Phi_q(k) = 1 - \frac{(1-\beta_1)(1-\beta_2)\gamma\left([k]_q-1\right)(1+\mu)}{\left[1-\beta_1+\gamma\left([k]_q-1\right)(1+\mu)\right]\left[1-\beta_2+\gamma\left([k]_q-1\right)(1+\mu)\right]\left[1+\left([k]_q-1\right)\delta\right]\Psi_{k,n,q}(\alpha,\lambda)-(1-\beta_1)(1-\beta_2)}, \tag{3.15}$$

since $\Phi_q(k)$ is an increasing function of $(k \geq 2)$, letting $k = 2$ in (3.15), we obtain

$$\rho_1 \leq \Phi_q(2) = 1 - \frac{\gamma q(1+\mu)(1-\beta_1)(1-\beta_2)}{[1-\beta_1+\gamma q(1+\mu)][1-\beta_2+\gamma q(1+\mu)](1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)-(1-\beta_1)(1-\beta_2)}. \tag{3.16}$$

This completes the proof.

Theorem 5. *If $f_j(z) \in TS_{\lambda,q}^{n,\alpha}(\delta,\gamma,\mu,\beta_j)$, $j = (1, 2, ..., p)$ is defined by (3.1), then $(f_1 * f_2 * ... * f_p)(z) \in TS_{\lambda,q}^{n,\alpha}(\delta,\gamma,\mu,\rho_2)$, where*

$$\rho_2 = 1 - \frac{\gamma q(1+\mu)\prod_{j=1}^{p}(1-\beta_j)}{\prod_{j=1}^{p}[1-\beta_j+\gamma q(1+\mu)]\left[(1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)\right]^{p-1} - \prod_{j=1}^{p}(1-\beta_j)}. \tag{3.17}$$

Proof. For $p = 1$, we see that $\rho_2 = \beta_1$. For $p = 2$, proceeding as Theorem 3.2, the result is true. Suppose that the result is true for any positive integer p. We must show that

$$(f_1 * f_2 * ... * f_{p+1})(z) \in TS_{\lambda,q}^{n,\alpha}(\delta,\gamma,\mu,\eta),$$

where

$$\eta = 1 - \frac{\gamma q(1+\mu)(1-\rho_2)(1-\beta_{p+1})}{[1-\rho_2+\gamma q(1+\mu)][1-\beta_{p+1}+\gamma q(1+\mu)](1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)-(1-\rho_2)(1-\beta_{p+1})} \tag{3.18}$$

and ρ_2 is given by (3.17). After simple compulations, we have

$$\eta = 1 - \frac{\gamma q(1+\mu)\prod_{j=1}^{p+1}(1-\beta_j)}{\prod_{j=1}^{p+1}[1-\beta_j+\gamma q(1+\mu)][(1+\delta q)\Psi_{2,n,q}(\alpha,\lambda)]^{p} - \prod_{j=1}^{p+1}(1-\beta_j)},$$

this shows that the result is true for $p + 1$. Therefore, by mathematical induction, the result is true for any positive integer p.

4. Partial sums

For $f \in S_{\lambda,q}^{n,\alpha}(\delta,\gamma,\mu,\beta)$, given by (1.1), its sequence of partial sums is

$$f_m(z) = z + \sum_{k=2}^{m} a_k z^k \quad (m \in \mathbb{N}). \tag{4.1}$$

Applying the technique used by ([5], [14], [16] and [22]) on partial sums of analytic univalent functions, to obtain the results. And let

$$B_k(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta) = \left[1-\beta+\gamma\left([k]_q-1\right)(1+\mu)\right]\left[1+\left([k]_q-1\right)\delta\right]\Psi_{k,n,q}(\alpha,\lambda). \tag{4.2}$$

Theorem 6. *Let $f \in S_{\lambda,q}^{n,\alpha}(\delta,\gamma,\mu,\beta)$, satisfies the condition (2.1), then*

$$(a)\ \ Re\left(\frac{f(z)}{f_m(z)}\right) \geq \frac{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)-(1-\beta)}{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}, \tag{4.3}$$

$$(b)\ \ Re\left(\frac{f_m(z)}{f(z)}\right) \geq \frac{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)+(1-\beta)}, \tag{4.4}$$

where

$$B_k(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta) \geq \begin{cases} 1-\beta & k=2,3,...,m \\ B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta) & k=m+1,.... \end{cases} \tag{4.5}$$

The result (4.3) and (4.4) are sharp with the function given by

$$f(z) = z + \frac{1-\beta}{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)} z^{m+1}. \tag{4.6}$$

Proof. Let

$$\frac{1+w(z)}{1-w(z)} = \frac{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}{1-\beta}\left[\frac{f(z)}{f_m(z)} - \frac{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)-(1-\beta)}{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}\right]$$

$$= \frac{1+\sum_{k=2}^{m} a_k z^{k-1} + \left(\frac{B_{m+1(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}}{1-\beta}\right)\sum_{k=m+1}^{\infty} a_k z^{k-1}}{1+\sum_{k=2}^{\infty} a_k z^{k-1}}. \tag{4.7}$$

It suffices to show that $|w(z)| \leq 1$. Now from (4.7) we have

$$w(z) = \frac{\left(\frac{B_{m+1(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}}{1-\beta}\right)\sum_{k=m+1}^{\infty} a_k z^{k-1}}{2+2\sum_{k=2}^{m} a_k z^{k-1} + \left(\frac{B_{m+1(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}}{1-\beta}\right)\sum_{k=m+1}^{\infty} a_k z^{k-1}}.$$

Hence

$$|w(z)| \leq \frac{\left(\frac{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}{1-\beta}\right)\sum_{k=m+1}^{\infty}|a_k|}{2-2\sum_{k=2}^{m}|a_k|-\left(\frac{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}{1-\beta}\right)\sum_{k=m+1}^{\infty}|a_k|}.$$

Now $|w(z)| \leq 1$ if

$$\sum_{k=2}^{m}|a_k| + \left(\frac{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}{1-\beta}\right)\sum_{k=m+1}^{\infty}|a_k| \leq 1.$$

From (2.1) it is sufficient to show that

$$\sum_{k=2}^{m}|a_k| + \left(\frac{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}{1-\beta}\right)\sum_{k=m+1}^{\infty}|a_k| \leq \sum_{k=2}^{\infty}\left(\frac{B_k(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}{1-\beta}\right)|a_k|,$$

which is equivalent to

$$\sum_{k=2}^{m}\left(\frac{B_k-1+\beta}{1-\beta}\right)|a_k| + \sum_{k=m+1}^{\infty}\left(\frac{B_k(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)-B_{m+1(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}}{1-\beta}\right)|a_k| \geq 0. \tag{4.8}$$

For $z = re^{i\pi/m}$ we have

$$\begin{aligned}\frac{f(z)}{f_m(z)} &= 1+\frac{1-\beta}{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}z^m \to 1-\frac{1-\beta}{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)} \\ &= \frac{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)-1+\beta}{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)} \text{ where } r \to 1^-,\end{aligned}$$

which shows that $f(z)$ given by (4.6) gives the sharpness.To prove the second part of this theorem, we write

$$\begin{aligned}\frac{1+w(z)}{1-w(z)} &= \frac{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)+1-\beta}{1-\beta} \\ &\times \left[\frac{f_m(z)}{f(z)} - \frac{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)+(1-\beta)}\right] \\ &= \frac{1+\sum_{k=2}^{m}a_k z^{k-1}-\left(\frac{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}{1-\beta}\right)\sum_{k=m+1}^{\infty}a_k z^{k-1}}{1+\sum_{k=2}^{m}a_k z^{k-1}},\end{aligned}$$

where

$$|w(z)| \leq \frac{\left(\frac{1-\beta+B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}{1-\beta}\right)\sum_{k=m+1}^{\infty}|a_k|}{2-2\sum_{k=2}^{m}|a_k| - \left(\frac{1-\beta-B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}{1-\beta}\right)\sum_{k=m+1}^{\infty}|a_k|}.$$

Now $|w(z)| \leq 1$ if

$$\sum_{k=2}^{m}|a_k| + \left(\frac{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}{1-\beta}\right)\sum_{k=m+1}^{\infty}|a_k| \leq 1.$$

Making use of (2.1) to get (4.8). Finally, equality holds in (4.4) for the external function f(z) given by (4.6).

Theorem 7. *Let* $f \in S_{\lambda,q}^{n,\alpha}(\delta,\gamma,\mu,\beta)$, *satisfies the condition* (2.1), *then*

$$(a)\ Re\left(\frac{f'(z)}{f'_m(z)}\right) \geq \frac{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)-(m+1)(1-\beta)}{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}, \tag{4.10}$$

$$(b)\ Re\left(\frac{f'_m(z)}{f'(z)}\right) \geq \frac{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)+(m+1)(1-\beta)}, \tag{4.11}$$

where

$$B_{k+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta) \geq \begin{cases} k(1-\beta) & k=2,3,...,m \\ k(1-\beta) + \frac{k(B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta))}{(m+1)} & k=m+1,.... \end{cases} \tag{4.12}$$

The results are sharp with the function given by (4.6).

Proof. We prove (a). The proof of (b) is similar and will be omitted.We write

$$\begin{aligned} \frac{1+w(z)}{1-w(z)} &= \frac{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}{(m+1)(1-\beta)} \\ &\times \left[\frac{f'(z)}{f'_m(z)} - \left(\frac{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)-(m+1)(1-\beta)}{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}\right)\right], \end{aligned}$$

where

$$w(z) = \frac{\left(\frac{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}{(m+1)(1-\beta)}\right)\sum_{k=m+1}^{\infty}ka_kz^{k-1}}{2+2\sum_{k=2}^{m}ka_kz^{k-1} + \left(\frac{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}{(m+1)(1-\beta)}\right)\sum_{k=m+1}^{\infty}ka_kz^{k-1}}.$$

Now $|w(z)| \leq 1$ if

$$\sum_{k=2}^{m} k|a_k| + \left(\frac{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}{(m+1)(1-\beta)}\right)\sum_{k=m+1}^{\infty} k|a_k| \leq 1.$$

From (2.1) it is sufficient to show that

$$\sum_{k=2}^{m} k|a_k| + \left(\frac{B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}{(m+1)(1-\beta)}\right)\sum_{k=m+1}^{\infty} k|a_k|$$
$$\leq \sum_{k=2}^{\infty}\left(\frac{B_k(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)}{1-\beta}\right)|a_k|,$$

which is equivalent to

$$\sum_{k=2}^{m}\left(\frac{B_k(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)-(1+\beta)k}{1-\beta}\right)|a_k| +$$

$$\sum_{k=m+1}^{\infty}\left(\frac{(m+1)B_k(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta)-(B_{m+1}(n,q,\alpha,\lambda,\delta,\gamma,\mu,\beta))k}{(m+1)(1-\beta)}\right)|a_k| \geq 0,$$

this completes the proof.

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