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AN APPROACH TO SYMMETRIC NUMERICAL SEMIGROUPS

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Abstract. In this study, we will get some results in a class of the family of symmetric numerical semigroups such that $S_r = \langle 7, 7r+6 \rangle$ where $r \geq 1$, $r \in \mathbb{Z}$. We will also examine Arf closure of these numerical semigroups.

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1. Introduction

Let $\mathbb{N}_0$ denote the monoid of nonnegative integers under addition. A submonoid S of $\mathbb{N}_0$ is called a numerical semigroup such that $\mathbb{N} \setminus S$ is finite. Let $S = \langle c_1, c_2, \cdots, c_n \rangle$ be numerical semigroup where $c_1, c_2, \cdots, c_n$ are relatively prime positive integer. In this case, we write

$$S = \langle c_1, c_2, \cdots, c_n \rangle = \left\{ \sum_{i=1}^{n} t_i c_i \, : \, t_i \in \mathbb{N}_0 \right\}.$$

Here, c_1 is called multiplicity of S and is denoted by $m(S)$. Let S be a numerical semigroup. Then, the greatest integer which doesn't belong to S is called the Frobenius number of S and is denoted by $F(S)$, that is $F(S) = max(\mathbb{Z} \backslash S)$. Also $n(S) = Card\left(\{0, 1, 2, \cdots, F(S)\} \cap S\right)$ is called determine number of S (see [5]). Thus we can write that

$$\begin{aligned} S = \langle c_1, c_2, \cdots, c_n \rangle &= \left\{ \sum_{i=1}^{n} t_i c_i \, : \, t_i \in \mathbb{N}_0 \right\} \\ &= \{ s_0 = 0, s_1, s_2, \cdots, s_{n-1}, s_n = F(S) + 1, \rightarrow \cdots \}, \end{aligned}$$

where $s_i < s_{i+1}$, $n = n(S)$ and the arrow means that every integer greater than $F(S) + 1$ belongs to S for $i = 1, 2, \cdots, n = n(S)$ (see [3]).

If $d \in \mathbb{N}_0$ and $d \notin S$, then d is called gap of S. We denote the set of gaps of S by $H(S)$, that is $H(S) = \{a \in \mathbb{N}_0 : a \notin S\}$ and the $G(S) = Card(H(S))$ is called the genus of S. It is known that $G(S) + n(S) = F(S) + 1$ (see [4]).

S is called symmetric numerical semigroup if $F(S) - p \in S$, for $p \in \mathbb{Z} \backslash S$. It is known the numerical semigroup $S = \langle c_1, c_2 \rangle$ is symmetric $F(S) = c_1 c_2 - c_1 - c_2$ and $n(S) = \frac{F(S)+1}{2}$ (see [1]).

A numerical semigroup S is called Arf if $c_1 + c_2 - c_3 \in S$, for all $c_1, c_2, c_3 \in S$ such that $c_1 \geq c_2 \geq c_3$. The smallest Arf numerical semigroup containing a numerical semigroup S is called the Arf closure of S, and is denoted by $Arf(S)$ (for details see [2, 6]). If S is a numerical semigroup such that $S = \langle c_1, c_2, \cdots, c_n \rangle$, then $L(S) = \langle c_1, c_2 - c_1, c_3 - c_1, \cdots, c_n - c_1 \rangle$ is called Lipman numerical semigroup of S, and it is known that

$$L_0(S) = S \subseteq L_1(S) = L(L_0(S)) \subseteq L_2 = L(L_1(S)) \subseteq \cdots \subseteq L_v = L(L_{v-1}(S)) \subseteq \cdots \subseteq \mathbb{N}.$$

In this study, we will show outcome of a class of symmetric numerical semigroups such that $S_r = \langle 7, 7r + 6 \rangle$ where $r \geq 1, r \in \mathbb{Z}$. Also, we will examine Arf closure of these numerical semigroups.

2. Main Results

Theorem 1. *Let $S_r = \langle 7, 7r + 6 \rangle$ be numerical semigroup where $r \geq 1, r \in \mathbb{Z}$. Then we have*

$$(a) F(S_r) = 42r + 29$$
$$(b) n(S_r) = 21r + 15$$
$$(c) G(S_r) = 21r + 15.$$

Proof. Let $S_r = \langle 7, 7r + 6 \rangle$ be numerical semigroup where $r \geq 1, r \in \mathbb{Z}$. Then S_r is symmetric and we find that

$$(a) F(S_r) = 7(7r + 6) - 7 - 7r - 6 = 42r + 29$$
$$(b) n(S_r) = \frac{F(S_r) + 1}{2} = \frac{42r + 29 + 1}{2} = 21r + 15$$
$$(c) G(S_r) = 42r + 29 + 1 - 21r - 15 = 21r + 15 \text{ from } G(S_r) = F(S_r) + 1 - n(S_r).$$

Theorem 2. *Let $S_r = \langle 7, 7r + 6 \rangle$ be numerical semigroup where $r \geq 1, r \in \mathbb{Z}$. Then $Arf(S_r) = \{0, 7, 14, 21, \cdots, 7r, 7r + 6, \rightarrow \cdots\}$.*

Proof. It is trivial $m_0 = 7$ since $L_0(S_r) = S_r$. Thus, we write $L_1(S_r) = \langle 7, 7r-1\rangle$. In this case,
(1) If $7r - 1 < 7$ (if $r = 1$) then $S_1 = \langle 7, 13\rangle$, $L_1(S_1) = \langle 7, 6\rangle = \langle 6, 7\rangle$, $m_1(S_1) = m_1 = 6$, $L_2(S_1) = \langle 6, 1\rangle = \langle 1, 6\rangle = \langle 1\rangle = \mathbb{N}_0$, $m_2(S_1) = m_2 = 1$.

In this way, we have that $Arf(S_1) = \{0, 7, 13, \rightarrow \cdots\}$.
(2) If $7r - 1 \geq 7$ (if $r \geq 2$) then $L_1(S_r) = \langle 7, 7r-1\rangle$ and $m_1(S_r) = m_1 = 7$. In this case, we write $L_2(S_r) = \langle 7, 7r-8\rangle$.
(a) If $r = 2$ then $L_2(S_2) = \langle 7, 6\rangle = \langle 6, 7\rangle$, $m_2(S_2) = m_2 = 6$, $L_3(S_2) = \langle 6, 1\rangle = \langle 1, 6\rangle = \langle 1\rangle = \mathbb{N}_0$, $m_3(S_2) = m_3 = 1$. So, we have $Arf(S_2) = \{0, 7, 14, 20, \rightarrow \cdots\}$.
(b) If $r > 2$ then $L_2(S_r) = \langle 7, 7r-8\rangle$ and $m_2(S_r) = m_2 = 7$ and $L_3(S_r) = \langle 7, 7r-15\rangle$. In this condition,
(i) If $r = 3$ then $L_3(S_3) = \langle 7, 6\rangle = \langle 6, 7\rangle$, $m_3(S_3) = m_3 = 6$, $L_4(S_3) = \langle 6, 1\rangle = \langle 1, 6\rangle = \langle 1\rangle = \mathbb{N}_0$, $m_4(S_3) = m_4 = 1$. So we find that $Arf(S_3) = \{0, 7, 14, 21, 27 \rightarrow \cdots\}$.
(ii) If $r > 3$ then $L_3(S_r) = \langle 7, 7r-15\rangle$ and $m_3(S_r) = m_3 = 7$ and $L_4(S_r) = \langle 7, 7r-22\rangle$. In this case,
(1')If $r = 4$ then $L_4(S_4) = \langle 7, 6\rangle = \langle 6, 7\rangle$, $m_4(S_4) = m_4 = 6$, $L_5(S_4) = \langle 6, 1\rangle = \langle 1, 6\rangle = \langle 1\rangle = \mathbb{N}_0$, $m_5(S_4) = m_5 = 1$. Thus we have $Arf(S_4) = \{0, 7, 14, 21, 28, 34 \rightarrow \cdots\}$.
(2')If $r > 4$ then $L_4(S_r) = \langle 7, 7r-22\rangle$ and $m_4(S_r) = m_4 = 7$ and we write $L_5(S_r) = \langle 7, 7r-29\rangle$. If we go on the operations then we obtain Arf closure of $Arf(S_r)$ as follows

$$Arf(S_r) = \{0, 7, 14, 21, \cdots, 7r, 7r+6, \rightarrow \cdots\}.$$

Thus, the proof is completed.

Corollary 3. *Let $S_r = \langle 7, 7r+6\rangle$ be numerical semigroup where $r \geq 1, r \in \mathbb{Z}$. Then we have*

$$(a) F(Arf(S_r)) = 7r + 5$$
$$(b) n(Arf(S_r)) = r + 1$$
$$(c) G(Arf(S_r)) = 6r + 5.$$

Proof. Let $S_r = \langle 7, 7r+6\rangle$ be numerical semigroup where $r \geq 1, r \in \mathbb{Z}$. So, we write that $F(Arf(S_r)) = 7r + 5$ from Theorem 2. On the other hand, we find that

$$n(Arf(S_r)) = Card(\{0, 1, 2, \cdots, 7r+5\} \cap Arf(S_r)) = Card(\{0, 7, 14, 21, \cdots, 7r\}) = r+1$$

and we obtain

$$G(Arf(S_r)) = 7r + 5 + 1 - r - 1 = 6r + 5$$

since

$$G(Arf(S_r)) = F(Arf(S_r)) + 1 - n(Arf(S_r)).$$

Corollary 4. *Let $S_r = \langle 7, 7r+6 \rangle$ be numerical semigroup where $r \geq 1, r \in \mathbb{Z}$. Then we have*

$$(a) F(S_r) = F(Arf(S_r)) + 35r + 24$$
$$(b) n(S_r) = n(Arf(S_r)) + 20r + 14$$
$$(c) G(S_r) = G(Arf(S_r)) + 15r + 10.$$

Proof. Let $S_r = \langle 7, 7r+6 \rangle$ be numerical semigroup where $r \geq 1, r \in \mathbb{Z}$. We have

$$(a) F(Arf(S_r)) + 35r + 24 = (7r+5) + 35r + 24 = 42r + 29 = F(S_r)$$
$$(b) n(Arf(S_r)) + 20r + 14 = (r+1) + 20r + 14 = 21r + 15 = n(S_r)$$
$$(c) G(Arf(S_r)) + 15r + 10 = (6r+5) + 15r + 10 = 21r + 15 = G(S_r)$$

from Corollary 3.

Corollary 5. *Let $S_r = \langle 7, 7r+6 \rangle$ be numerical semigroup where $r \geq 1, r \in \mathbb{Z}$. Then we have*

$$(a) F(S_{r+1}) = F(S_r) + 42$$
$$(b) n(S_{r+1}) = n(S_r) + 21$$
$$(c) G(S_{r+1}) = G(S_r) + 21.$$

Corollary 6. *Let $S_r = \langle 7, 7r+6 \rangle$ be numerical semigroup where $r \geq 1, r \in \mathbb{Z}$. Then the following equalities are satisfied:*

$$(a) F(Arf(S_{r+1})) = F(Arf(S_r)) + 7$$
$$(b) n(Arf(S_{r+1})) = n(Arf(S_r)) + 1$$
$$(c) G(Arf(S_{r+1})) = G(Arf(S_r)) + 6.$$

Example 1. *We put $r = 1$ in $S_r = \langle 7, 7r+6 \rangle$ symmetric numerical semigroup. Then we have*

$$S_1 = \langle 7, 13 \rangle = \{0, 7, 13, 14, 20, 21, 26, 27, 28, 33, 34, 35, 39, 40, 41, 42, \cdots, 72, \rightarrow, \cdots\}.$$

In this case, we have $F(S_1) = 71$,$n(S_1) = 36$,

$$H(S_1) = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 15, 16, 17, 18, 19, 22, 23, 24, 25, 29, 30, 31, 32, 36, 37, 38, 43, 45, 46, 50, 51, 58, 64, 65, 69, 71\},$$

$$G(S_1) = Card(H(S_1)) = 36$$

$$Arf(S_1) = \{0, 7, 13, \rightarrow, \cdots\},$$
$$F(Arf(S_1)) = 12,$$
$$n(Arf(S_1)) = 2,$$
$$H(Arf(S_1)) = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12\}$$

and

$$G(Arf(S_1)) = 11.$$

So we get

$$\begin{aligned} &F(Arf(S_1)) + 35 + 24 = 59 + 12 = 71 = F(S_1) \\ &n(Arf(S_1)) + 20 + 14 = 34 + 2 = 36 = n(S_1) \\ &G(Arf(S_1)) + 15 + 10 = 25 + 11 = 36 = G(S_1) \end{aligned}$$

from Corollary 4.

We put $r = 2$ *then we write in* $S_r = \langle 7, 7r + 6\rangle$. *Then we write*

$$S_2 = \langle 7, 20\rangle = \{0, 7, 14, 21, 27, \cdots, 114, \rightarrow \cdots\}.$$

We have

$$\begin{aligned} &F(S_2) = 113 \\ &n(S_2) = 57 \\ &G(S_2) = Card(H(S_2)) = 57 \\ &Arf(S_2) = \{0, 7, 14, 20, \rightarrow \cdots\} \\ &G(Arf(S_2)) = 17. \end{aligned}$$

So, we find that

$$\begin{aligned} &F(Arf(S_2)) + 70 + 24 = 94 + 19 = 113 = F(S_2) \\ &n(Arf(S_2)) + 40 + 14 = 54 + 3 = 57 = n(S_2) \\ &G(Arf(S_2)) + 30 + 10 = 40 + 17 = 57 = G(S_2) \end{aligned}$$

from Corollary 4.

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