

APPLICATION OF OPERATOR SPLITTING BY QUARTIC B-SPLINE COLLOCATION FOR BURGERS' EQUATION

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ABSTRACT. Some of the modern differential equations applications include fluid mechanics, population dynamics, convective transport, electrical networks, chemical reaction kinetics, and molecular dynamics. Differential equations modeling physical processes are often contained different mathematical terms in its structure. It is sometimes complicated and challenging to analyze such equations both theoretically and numerically. In this study, numerical solutions of Burgers' equation modeling many physical phenomena are obtained. For this purpose, the Burgers' equation is divided by time and converted into two simpler sub-problems. Quartic B-spline bases are used with the collocation method to solve each subproblem, and stability analysis is given for time splitting. The obtained numerical solutions have been compared with other studies in the literature. Thus it has been shown that the proposed method is effective, sensitive, and applicable for many nonlinear models.

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1. INTRODUCTION

With the increasing computing power of modern digital computers in recent years, partial differential equations (PDEs) have become highly successful tools describing almost every physical and engineering problem. In recent years, mathematicians have increasingly been interested in PDEs' solutions depending on location and time containing high-order small coefficient linear terms [1]. The Burgers' equation is also an important PDE representing various physical phenomena. The Burgers' equation was first introduced by [2]. In later years, Burgers' [3, 4] used this equation to study some of the properties of the turbulent fluid in a channel resulting from the interaction of the opposite effects of convection and diffusion. Furthermore, after that study of Burgers', the equation was called the Burgers' equation. Besides, due

to the similarity of the Burgers' equation to the Navier-Stokes equation, solving this equation numerically also sheds light on the Navier-Stokes equation [5]. The equation we will consider in this study, with the initial and boundary conditions are as follows

$$U_t + UU_x - \nu U_{xx} = 0, a \leq x \leq b, t \geq 0 \quad (1)$$

$$U(x, 0) = f(x), U(a, t) = \lambda_1, U(b, t) = \lambda_2 \quad (2)$$

$$U_x(a, t) = U_x(b, t) = 0, U_{xx}(a, t) = U_{xx}(b, t) = 0. \quad (3)$$

Where $U = U(x, t)$ is a function that is sufficiently differentiable by position and time, and ν is the kinematic viscosity coefficient that controls the balance between convection and diffusion. When $\nu = 0$, the equation (1) becomes the inviscid Burgers' equation, which is a model for equations that develop shock waves. The equation (1) contains the first-order derivative and the second-order derivative with respect to t and x , respectively. It has a structure similar to the parabolic heat equation besides the term nonlinear UU_x . In addition to being the fundamental equation of fluid mechanics, this equation also models shock waves, gas dynamics, nonlinear acoustics, traffic flow, turbulence, elastic waves in anisotropic solid and heat conduction phenomena [6, 7, 8]. The exact solution of the Burgers' equation is given in [1, 8] with the Hopf - Cole transform as a Fourier series expansion. Further, the exact solution of the one-dimensional Burgers' equation for different initial values was obtained by [9]. In recent years, numerical solutions of the equation (1) have been obtained by many authors via various methods and techniques. Xue and Feng [10] proposed an alternating segment explicit-implicit (ASE-I) scheme with intrinsic parallelism for Burgers' equation, Arora et al. [11] solved Burgers' equation using an innovative scheme of collocation having quintic Hermite splines as base functions, Cook et al. [12] presented semi-implicit semi-Lagrangian (SISL) finite difference methods to approximate travelling wave solutions of the one-dimensional Burgers' equation, Zhao et al. [13] proposed mixed finite volume element (MFVE) methods for solving Burgers' equation, Mous and Laouar [14] solved the equation via finite difference method combined with explicit and implicit schemes, Elgindy and Karasözen [15] presented a high-order integral nodal discontinuous Galerkin (DG) method to solve Burgers' equation, Zhang et al. [16] concerned with the numerical analysis of a nonlinear implicit difference scheme for Burgers' equation, Özis et al. [17] used Hopf-Cole transformation to solve Burgers' equation via Galerkin quadratic B-spline finite element method, Kutluay et al. [18] calculated the numerical solutions of the equation by least-squares quadratic B-spline finite element method, Chen and Zhang [5] proposed a weak Galerkin (WG) finite element method to obtain solutions of the Burgers' equation, Zeidan et al. [19] developed a novel Adomian decomposition method (ADM) for the solution of Burgers' equation, Korkmaz et al. [20] intro-

duced a new differential quadrature method based on quartic B-spline functions to get solutions of the Burgers' equation and so on. Recently, It can be given as studies where splitting methods are applied as follows, Seydaoğlu [21] proposed multiquadric radial basis function (MQ-RBF) for space approximation and a Lie-Group scheme for time integration to solve Burgers' equation, Sari et al. [22] solved the equation using some higher order splitting-up techniques based on the cubic B-spline Galerkin finite element method, Saka and Dağ [23] described collocation method using quartic B-splines to obtain numerical solutions of the Burgers' equation, Geiser et al. [24] proposed adaptive iterative splitting methods to solve Multiphysics problems, which are related to convection–diffusion–reaction equations, Çiçek and Tanoğlu [25] derived an analytical approach to the Strang splitting method for the Burgers'-Huxley equation, Zürnacı and Seydaoğlu [26] presented convergence analysis of operator splitting methods applied to the nonlinear Rosenau–Burgers' equation, Zhang et al. [27] introduced the nonlinear stability and convergence analyses for a second order operator splitting scheme applied to the “good” Boussinesq equation.

In this study, numerical solutions of one-dimensional Burgers' equation were obtained using the quartic B-spline collocation finite element method using operator splitting. For this purpose, the equation has been converted into two subproblems, one linear and one nonlinear. After making approaches to the derivatives in each sub-problem with quadratic B-spline bases, ordinary differential equation systems were obtained. Numerical solutions of the equation are obtained by solving these systems sequentially. Some advantages of operator splitting methods are easy to apply and explicit methods, their algorithms are sequential, and mid-range solutions are stored in the solution vector, they retain the structural features of the solution (volume preservation, time symmetricity, and simple implementation can be given), thus providing superiority over other standard integrators in numerical schemes. In this respect, splitting methods are an important class of geometric numerical integrators [28].

2. ALGORITHM FOR OPERATOR SPLITTING

The fundamental idea behind operator splitting methods, dividing a given complex problem into more straightforward problems for smaller time steps. Thus, different parts of the problem can be effectively solved by appropriate integration methods [29].

Let us consider a Cauchy problem given as follows

$$\frac{dU(t)}{dt} = \Lambda U(t), \quad U(0) = U_0, \quad t \in [0, T]. \quad (4)$$

In (4), it is assumed that the function $U(x, t)$ is semi-discretized along spatial direction. We will concentrate on cases where the operator $\Lambda = \hat{A} + \hat{B}$ can be written as a summation of two linear (and/or nonlinear) operators. That is, it can be written as follows

$$\frac{dU(t)}{dt} = \hat{A}(U(t)) + \hat{B}(U(t)), \quad U(0) = U_0, \quad t \in [0, T]. \quad (5)$$

The vector $U(x, t)$ is the solution vector obtained from the $U_0 \in X$ initial condition. The operators $\Lambda, \hat{A}, \hat{B}$ are bounded or unbounded operators in a finite or infinite X Banach space. With the aid of the Lie operator formulation, the expression (5) in general (maybe nonlinear) can be written as follows

$$\frac{dU(t)}{dt} = AU(t) + BU(t) \quad (6)$$

Where, the operators A and B are Lie operators applied to the function $U(t)$ like $A = \hat{A}(U(t)) \frac{\partial}{\partial U}$, $B = \hat{B}(U(t)) \frac{\partial}{\partial U}$. The formal solution of the problem (6) is $U(t_{n+1}) = e^{\Delta t(A+B)}U(t_n)$. This solution can also be written as follows using the Taylor series expansion of exponential function

$$U(t_{n+1}) = e^{\Delta t(A+B)}U(t_n) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(\hat{A}(U(t)) \frac{\partial}{\partial U} + \hat{B}(U(t)) \frac{\partial}{\partial U} \right)^k U(t_n).$$

In order to solve Eq. (6) numerically, the splitting technique splits the problem into $\frac{dU(t)}{dt} = AU(t)$ and $\frac{dU(t)}{dt} = BU(t)$ and tries to find the solution numerically or analitically [30]. Let us assume that $\varphi_{\Delta t}^{[A]}$ and $\varphi_{\Delta t}^{[B]}$ are the exact or numerical solutions of the equations involving the operators A and B , respectively, then the most basic first order splitting technique is defined as follows

$$L_{\Delta t} = \varphi_{\Delta t}^{[A]} \circ \varphi_{\Delta t}^{[B]} \equiv e^{\Delta t A} e^{\Delta t B} \quad \text{or} \quad L_{\Delta t}^* = \varphi_{\Delta t}^{[B]} \circ \varphi_{\Delta t}^{[A]} \equiv e^{\Delta t B} e^{\Delta t A}$$

and known as Lie-Trotter splitting [31] techniques. If we change the place of operators A and B and take the combination for half time step as follows

$$S_{\Delta t} = e^{\frac{\Delta t}{2} A} e^{\Delta t B} e^{\frac{\Delta t}{2} A} \quad \text{or} \quad S_{\Delta t}^* = e^{\frac{\Delta t}{2} B} e^{\Delta t A} e^{\frac{\Delta t}{2} B}$$

we obtain so-called symmetric Marchuk [32] or more widely known Strang splitting [33] techniques having the schemes "A – B – A" and "B – A – B". The algorithm for Strang splitting technique is as follows

$$\begin{aligned} \frac{dU^*(t)}{dt} &= AU^*(t), \quad U^*(t_n) = U_n^0, & t \in \left[t_n, t_{n+\frac{1}{2}} \right] \\ \frac{dU^{**}(t)}{dt} &= BU^{**}(t), \quad U^{**}(t_n) = U^*(t_{n+\frac{1}{2}}), & t \in [t_n, t_{n+1}] \\ \frac{dU^{***}(t)}{dt} &= AU^{***}(t), \quad U^{***}(t_{n+\frac{1}{2}}) = U^{**}(t_{n+1}), & t \in \left[t_{n+\frac{1}{2}}, t_{n+1} \right]. \end{aligned} \quad (7)$$

Here $t_{n+\frac{1}{2}} = t_n + \frac{\Delta t}{2}$ and the desired solutions are obtained from equation $U(t_{n+1}) = U^{***}(t_{n+1})$. Replacing the original problem with sub-problems naturally leads to an error called splitting error [29]. The local truncation error of this approach is found as follows

$$\begin{aligned} Te &= \frac{(e^{\Delta t(A+B)} - e^{\frac{\Delta t}{2}A}e^{\Delta tB}e^{\frac{\Delta t}{2}A})U(t_n)}{\Delta t} \\ &= \frac{\Delta t^2}{24}(2[B, [B, A]] - [A, [A, B]])U(t_n) + O(\Delta t^3). \end{aligned}$$

Moreover, this illustrates that the technique is the second-order one. Here the symbols $[,]$ are Lie parenthesis, and they represent the Baker-Campbell-Hausdorff (BCH) formula, which has been frequently used in numerical analysis in recent years. One can see the Ref. [34] and therein for more information. Also, Faou et al. [35] analyzed the convergence properties of the exponential Lie and Strang splitting applied to inhomogeneous second-order parabolic equations with homogeneous Dirichlet boundary conditions. Hansen and Ostermann [36] concerned an optimal convergence analysis is presented for the methods when applied to equations on Banach spaces with unbounded vector fields. Descombes [37] proved the convergence of a splitting scheme of high order for a reaction-diffusion system.

3. QUARTIC B-SPLINE COLLOCATION APPROACH

The partition of the solution interval $[a, b]$ in terms of nodal points x_m , $m = 0, 1, \dots, N$, is $a = x_0 < x_1 < \dots < x_N = b$. If $h = x_{m+1} - x_m$, $F_m(x)$, $m = -2(1)N + 1$, quartic B-spline functions on the range $[a, b]$ in terms of nodes x_m as follows

$$F_m(x) = \frac{1}{h^4} \begin{cases} (x - x_{m-2})^4, & [x_{m-2}, x_{m-1}] \\ (x - x_{m-2})^4 - 5(x - x_{m-1})^4, & [x_{m-1}, x_m] \\ (x - x_{m-2})^4 - 5(x - x_{m-1})^4 + 10(x - x_m)^4, & [x_m, x_{m+1}] \\ (x_{m+3} - x)^4 - 5(x_{m+2} - x)^4, & [x_{m+1}, x_{m+2}] \\ (x_{m+3} - x)^4, & [x_{m+2}, x_{m+3}] \\ 0, & \text{otherwise.} \end{cases} \quad (8)$$

It is clear that the set $\{F_{-2}(x), F_{-1}(x), \dots, F_{N+1}(x)\}$ forms a base on the interval $[a, b]$ [39]. It is clear that $F_m(x)$ function is zero outside the range $[x_{m-2}, x_{m+3}]$. So each $[x_m, x_{m+1}]$ finite element is covered by five quartic B-splines such as $F_{m-2}(x)$, $F_{m-1}(x)$, $F_m(x)$, $F_{m+1}(x)$, $F_{m+2}(x)$. Let's assume that the function $U(x, t)$ is defined on $[a, b]$, then, $U(x, t)$ can be approached in terms of quadratic B-spline

functions and time dependent $\delta_m(t)$ parameters as follows

$$U(x, t) \cong \sum_{-2}^{N+1} \delta_m(t) F_m(x). \quad (9)$$

Where $\delta_m(t)$ are time-dependent parameters determined in each time step. With the help of expressions (8) and (9), the values of $U(x, t)$ and its derivatives at the node points are as follows

$$\begin{aligned} U_m &= \delta_{m-2} + 11\delta_{m-1} + 11\delta_m + \delta_{m+1}, \\ U'_m &= \frac{4}{h}(-\delta_{m-2} - 3\delta_{m-1} + 3\delta_m + \delta_{m+1}), \\ U''_m &= \frac{12}{h^2}(\delta_{m-2} - \delta_{m-1} - \delta_m + \delta_{m+1}), \\ U'''_m &= \frac{24}{h^3}(-\delta_{m-2} + 3\delta_{m-1} - 3\delta_m + \delta_{m+1}). \end{aligned} \quad (10)$$

4. APPLICATION OF THE METHOD TO BURGERS' EQUATION

The Burgers' (1) equation is split as linear part (diffusion) and nonlinear part (convection) as follows

$$U_t - \nu U_{xx} = 0 \quad (11)$$

$$U_t + UU_x = 0. \quad (12)$$

If the approaches given in (10) are used in (11) and (12), ordinary differential equation systems are obtained as follows

$$\overset{\circ}{\delta}_{m-2} + 11\overset{\circ}{\delta}_{m-1} + 11\overset{\circ}{\delta}_m + \overset{\circ}{\delta}_{m+1} - \frac{12\nu}{h^2}(\delta_{m-2} - \delta_{m-1} - \delta_m + \delta_{m+1}) = 0 \quad (13)$$

$$\overset{\circ}{\delta}_{m-2} + 11\overset{\circ}{\delta}_{m-1} + 11\overset{\circ}{\delta}_m + \overset{\circ}{\delta}_{m+1} + \frac{4z_m}{h}(-\delta_{m-2} - 3\delta_{m-1} + 3\delta_m + \delta_{m+1}) = 0. \quad (14)$$

Where the symbol \circ denotes the first order derivative with respect to t and z_m is $z_m = \delta_{m-2} + 11\delta_{m-1} + 11\delta_m + \delta_{m+1}$. Instead of the parameter δ_m and $\overset{\circ}{\delta}_m$, $\frac{\delta_m^{n+1} + \delta_m^n}{2}$ Crank-Nicolson and $\frac{\delta_m^{n+1} - \delta_m^n}{\Delta t}$ forward finite difference approaches are written in the equations (13) and (14) respectively, the following algebraic equation systems are obtained

$$\kappa_1 \delta_{m-2}^{n+1} + \kappa_2 \delta_{m-1}^{n+1} + \kappa_2 \delta_m^{n+1} + \kappa_1 \delta_{m+1}^{n+1} = \kappa_3 \delta_{m-2}^n + \kappa_4 \delta_{m-1}^n + \kappa_4 \delta_m^n + \kappa_3 \delta_{m+1}^n \quad (15)$$

$$\kappa_5 \delta_{m-2}^{n+1} + \kappa_6 \delta_{m-1}^{n+1} + \kappa_7 \delta_m^{n+1} + \kappa_8 \delta_{m+1}^{n+1} = \kappa_8 \delta_{m-2}^n + \kappa_7 \delta_{m-1}^n + \kappa_6 \delta_m^n + \kappa_5 \delta_{m+1}^n \quad (16)$$

$$\begin{aligned}\kappa_1 &= 1 - \frac{6\nu\Delta t}{h^2}, \quad \kappa_2 = 11 + \frac{6\nu\Delta t}{h^2}, \quad \kappa_3 = 1 + \frac{6\nu\Delta t}{h^2}, \quad \kappa_4 = 11 - \frac{6\nu\Delta t}{h^2}, \\ \kappa_5 &= 1 - \frac{2z_m\Delta t}{h}, \quad \kappa_6 = 11 - \frac{6z_m\Delta t}{h}, \quad \kappa_7 = 11 + \frac{6z_m\Delta t}{h}, \quad \kappa_8 = 1 + \frac{2z_m\Delta t}{h}.\end{aligned}$$

(15) and (16) systems consist of $(N + 1)$ equations and $(N + 4)$ unknown parameters $(\delta_{-2}, \delta_{-1}, \delta_0, \dots, \delta_N, \delta_{N+1})$. For the unique solution of these systems, δ_{-2} , δ_{-1} and δ_{N+1} values must be eliminated. If the boundary conditions given by (2) and (3) are used to eliminate these parameters, consequently one obtained four-diagonal band matrix systems of $(N + 1) \times (N + 1)$. In order to obtain numerical solutions in the desired time step, initial vector is calculated first. Then, (15) and (16) systems are resolved with the Strang splitting algorithm (7) using the initial vector. The initial vector δ_m^0 is obtained using the $U(x, 0) = f(x)$ as follows

$$\delta_{m-2}^0 + 11\delta_{m-1}^0 + 11\delta_m^0 + \delta_{m+1}^0 = f(x_m), \quad m = 0(1)N.$$

To be able to solve this system uniquely, the parameters δ_{-2} , δ_{-1} and δ_{N+1} must be eliminated using boundary conditions $U_x(a, 0) = U_{xx}(a, 0) = 0$ and $U_x(b, 0) = 0$. Thus, four-diagonal band matrix system of $(N + 1) \times (N + 1)$ has been obtained as follows

$$\begin{bmatrix} 18 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 11.5 & 11.5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 11 & 11 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 11 & 11 & 1 & 0 & 0 & 0 \\ & & & \ddots & & & & \\ 0 & 0 & 0 & 1 & 11 & 11 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 11 & 11 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 14 & 8 \end{bmatrix} \begin{bmatrix} \delta_0^0 \\ \delta_1^0 \\ \delta_2^0 \\ \delta_3^0 \\ \vdots \\ \delta_{N-2}^0 \\ \delta_{N-1}^0 \\ \delta_N^0 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_{N-2}) \\ f(x_{N-1}) \\ f(x_N) \end{bmatrix}$$

4.1. Stability Analysis

To investigate the stability analysis of the proposed method, let us express the stability of the systems (15) and (16) with ρ_A and ρ_B , respectively. For this purpose, the von Neumann method [38] will be applied in accordance with the algorithm (7). To do this, if the necessary operations are performed after typing $\delta_m^n = \xi^n e^{i\beta mh}$ in

the system (15) the following statement is obtained

$$\begin{aligned}\rho_A\left(\frac{\xi^{n+1/2}}{\xi^n}\right) &= \frac{\kappa_3 \cos 2\beta h + (\kappa_3 + \kappa_4) \cos \beta h + \kappa_4 + i[(\kappa_3 - \kappa_4) \sin \beta h - \kappa_3 \sin 2\beta h]}{\kappa_1 \cos 2\beta h + (\kappa_1 + \kappa_2) \cos \beta h + \kappa_2 + i[(\kappa_1 - \kappa_2) \sin \beta h - \kappa_1 \sin 2\beta h]} \\ \rho_A\left(\frac{\xi^{n+1/2}}{\xi^n}\right) &= \frac{X + iY}{Z + iT} \\ X &= \kappa_3 \cos 2\beta h + (\kappa_3 + \kappa_4) \cos \beta h + \kappa_4, Y = (\kappa_3 - \kappa_4) \sin \beta h - \kappa_3 \sin 2\beta h \\ Z &= \kappa_1 \cos 2\beta h + (\kappa_1 + \kappa_2) \cos \beta h + \kappa_2, T = (\kappa_1 - \kappa_2) \sin \beta h - \kappa_1 \sin 2\beta h.\end{aligned}$$

A method is stable if $\left|\rho_A\left(\frac{\xi^{n+1/2}}{\xi^n}\right)\right| \leq 1$. Since $|Z|^2 + |T|^2 - |X|^2 - |Y|^2 = 192v\Delta t[\cos \beta h - 5 \cos^2 \beta h - \cos^3 \beta h + 5]/h^2 \geq 0$, $\left|\rho_A\left(\frac{\xi^{n+1/2}}{\xi^n}\right)\right| \leq 1$ is provided. Since UU_x is linearized in the equation (12) z_m acts as a constant. If $\delta_m^n = \xi^n e^{i\beta m h}$ is written in the system (16) the following statement is obtained as a result of the necessary operations

$$\begin{aligned}\rho_B\left(\frac{\xi^{n+1}}{\xi^n}\right) &= \frac{\kappa_8 \cos 2\beta h + (\kappa_5 + \kappa_7) \cos \beta h + \kappa_6 + i[(\kappa_5 - \kappa_7) \sin \beta h - \kappa_8 \sin 2\beta h]}{\kappa_5 \cos 2\beta h + (\kappa_6 + \kappa_8) \cos \beta h + \kappa_7 + i[(\kappa_8 - \kappa_6) \sin \beta h - \kappa_5 \sin 2\beta h]} \\ \rho_B\left(\frac{\xi^{n+1}}{\xi^n}\right) &= \frac{K + iL}{M + iN} \\ K &= \kappa_8 \cos 2\beta h + (\kappa_5 + \kappa_7) \cos \beta h + \kappa_6, L = (\kappa_5 - \kappa_7) \sin \beta h - \kappa_8 \sin 2\beta h \\ M &= \kappa_5 \cos 2\beta h + (\kappa_6 + \kappa_8) \cos \beta h + \kappa_7, N = (\kappa_8 - \kappa_6) \sin \beta h - \kappa_5 \sin 2\beta h.\end{aligned}$$

Since here $|M|^2 + |N|^2 - |K|^2 - |L|^2 \geq 0$, $\left|\rho_B\left(\frac{\xi^{n+1}}{\xi^n}\right)\right| \leq 1$ is provided. So the Strang algorithm given with (7) is unconditionally stable since

$$\begin{aligned}\rho(\xi) &= \rho_A^{n+1/2} \rho_B^{n+1} \rho_A^{n+1/2} \\ |\rho(\xi)| &= \left|\rho_A\left(\frac{\xi^{n+1/2}}{\xi^n}\right)\right| \left|\rho_B\left(\frac{\xi^{n+1}}{\xi^n}\right)\right| \left|\rho_A\left(\frac{\xi^{n+1/2}}{\xi^n}\right)\right| \leq 1.\end{aligned}$$

5. APPLICATIONS AND RESULTS

In order to measure the effectiveness of the proposed method in this section, three test problems are taken into consideration, and the error norms L_2 , L_∞ and $\|e\|_1$ given below are used

$$L_2 = \sqrt{h \sum_{j=0}^N \left| (U_j - U_j^e)^2 \right|}, L_\infty = \max_j |U_j - U_j^e|, \|e\|_1 = \frac{1}{N} \sum_1^{N-1} \frac{|U_j - U_j^e|}{U_j^e}.$$

Table 1: Comparison of the $\|e\|_1$ norm at $t = 0.1$ for the values $\Delta t = 0.0005$, $\nu = 1$ and the different mesh sizes for the Problem 1.

Present method $\Delta t = 0.0005$	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.0125$	$h = 0.00625$
$L_2 \times 10^3$	0.361987	0.052935	0.005775	0.024239	0.014925
$L_\infty \times 10^3$	0.599086	0.088908	0.009633	0.039417	0.046720
$\ e\ _1$	0.0013363	0.0002235	0.0000316	0.0000931	0.0000920
Other methods $\ e\ _1$					
$\Delta t = 0.00001$					
[18] Least-squares	0.012165	0.006941	0.003651	0.001858	0.000928
[23] QBCM1	0.00174	0.00029	0.00005	0.00014	0.00016
[23] QBCM2-split	0.00177	0.00041	0.00006	0.00013	0.00016
[40] Galerkin- split	0.000635	0.000085	0.000024	0.000006	0.000006
[41] Cub-col.	0.00734	0.00095	0.00014	0.00003	0.00001
[42] Finite diff.	0.007571	0.002025	0.000555	0.000177	–

Problem 1

This example has the initial condition $U(x, 0) = \sin \pi x$ and the boundary conditions $U(0, t) = U(1, t) = 0$ that are frequently used in the literature, and its exact solution is given by Cole [8] as follows

$$\begin{aligned}
 U^e(x, t) &= 2\pi v \frac{\sum_{j=1}^{\infty} j a_j \sin(j\pi x) e^{-j^2 \pi^2 v t}}{a_0 + \sum_{j=1}^{\infty} a_j \cos(j\pi x) e^{-j^2 \pi^2 v t}} \\
 a_0 &= \int_0^1 e^{-(2\pi v)^{-1}(1-\cos \pi x)} dx \\
 a_j &= 2 \int_0^1 e^{-(2\pi v)^{-1}(1-\cos \pi x)} \cos(j\pi x) dx, \quad j = 1, 2, \dots
 \end{aligned}$$

In Table 1, the error norms of L_2 , L_∞ and $\|e\|_1$ at $t = 0.1$ are given of Problem 1 for the $\nu = 1$, $\Delta t = 0.0005$ and decreasing of the values h . It is seen that from the Table, the proposed splitting method gives better results than those given in Ref. [23] despite using the same bases and methods. Also, generally better results were obtained from the other studies given in the Table except for the Ref. [40] study using the Galerkin method. Figure 1 shows that the exact solution and numerical solution are in good harmony for the values $\nu = 1, 0.1$, and 0.01 . The numerical solutions for $\nu = 10^{-3}, 10^{-4}$ and 10^{-5} were plotted at different times because the exact solution was broken for the smaller values ν . L_2 and L_∞ error norms are given in Table 2 and the L_∞ norm is compared with some studies in the literature. As seen from the Table, we have a smaller L_∞ norm than the QBCM2 (split) method given in Ref. [23], when we compare with the QBCM1 method, while our norm of L_∞ is big in the early times, it is getting smaller as time progresses. In addition, in Table 3, positions and values where the solution has the maximum values are given.

Table 2: Comparison of the L_∞ norm for the values $h = 0.05$, $\Delta t = 0.01$ for $\nu = 1$, $h = \Delta t = 0.025$ for $\nu = 0.1$, $h = \Delta t = 0.01$ for $\nu = 0.01$.

t	ν	Present		[23]	[43]	[44]	[45]		
		$L_2 \times 10^3$	$L_\infty \times 10^3$	QBCM1 L_∞	QBCM2 L_∞	L_∞	L_∞	$N = 25$ L_∞	$N = 45$ L_∞
0.02	1	1.5453	2.2470	$2.9e-4$	$2.34e-3$	$5.19e-3$	-	-	-
0.04		1.2756	2.5098	$3.8e-4$	$2.61e-3$	$6.91e-3$	-	-	-
0.1		0.0869	0.1381	$3.7e-4$	$1.46e-3$	$8.17e-3$	-	-	-
0.22		0.0445	0.0698	$2.2e-4$	$3.60e-4$	$5.50e-3$	-	-	-
0.05	0.1	0.4226	1.1311	$8.7e-4$	$3.07e-3$	$3.98e-3$	$6.14e-4$	-	-
0.25		0.3340	0.7415	$1.2e-4$	$1.16e-2$	$9.03e-3$	$7.63e-4$	-	-
0.75		0.1085	0.2104	$4.0e-5$	$6.64e-3$	$3.91e-3$	$1.66e-4$	-	-
1.5		0.0130	0.0243	$3.0e-5$	$1.48e-3$	$1.25e-3$	$7.70e-5$	-	-
0.4	0.01	0.4701	3.5066	$8.8e-3$	$1.21e-2$	$2.60e-2$	$3.22e-3$	$1.6e-2$	$1.1e-3$
0.8		0.6045	1.9625	$1.5e-4$	$5.66e-3$	$2.88e-2$	$5.98e-3$	$2.6e-2$	$1.4e-3$
1.2		0.1020	0.4741	$8.0e-5$	$8.13e-3$	$1.77e-2$	$1.29e-3$	$8.0e-3$	$1.9e-4$
3		0.0039	0.0098	$1.0e-5$	$2.55e-3$	$6.93e-3$	$2.57e-5$	$4.5e-5$	$2.2e-5$

Table 3: Positions and values where the waves take their maximum value.

ν	t	x	$U(x, t)$	$U^e(x, t)$	ν	t	x	$U(x, t)$
1	0	0.5000	1.000000	1.000000	10^{-3}	0	0.5000	1.000000
	0.02	0.5000	0.821443	0.820095		0.2	0.7000	0.998071
	0.04	0.5000	0.672662	0.672485		0.4	0.9000	0.996191
	0.1	0.5000	0.371405	0.371577		0.8	0.9900	0.848825
	0.22	0.5000	0.113582	0.113668		1.2	0.9800	0.636106
0.1	0	0.5000	1.000000	1.000000	2	0.9800	0.423046	
	0.05	0.5500	0.951726	0.951723	0.2	0.7000	0.999821	
	0.25	0.6500	0.775191	0.775103	0.4	0.9000	0.999852	
	0.75	0.6500	0.426362	0.426330	10^{-4}	0.6	0.9900	1.003239
	1.5	0.5750	0.183364	0.183363	1	0.9900	0.759597	
0.01	0	0.5000	1.000000	1.000000	2	0.9900	0.440444	
	0.4	0.8900	0.956783	0.956557	0.2	0.7000	0.999950	
	0.8	0.9400	0.782100	0.781193	0.4	0.9000	0.999997	
	1.2	0.9200	0.584944	0.584724	10^{-5}	0.6	0.9990	1.096807
	3	0.8700	0.267775	0.267766	1	0.9990	0.828144	
					2	0.9990	0.469876	

Table 4: Comparison of the L_∞ norm with Ref. [13, 16] for the different values h and Δt .

$h, \Delta t$	Present			[13]
	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_\infty(U - U_h)$	$L_\infty(P - P_h)$
1/20, 1/20	0.2976	1.0451	2.6767e-2	2.1361e-2
1/40, 1/40	0.0732	0.2955	1.4241e-2	1.1424e-2
1/80, 1/80	0.0229	0.0783	7.4366e-3	5.9368e-3
1/160, 1/160	0.0073	0.0201	3.8548e-3	3.0320e-3

	Present		[16]
	$L_2 \times 10^7$	$L_\infty \times 10^7$	L_∞
1/20, 1/1600	0.312053	0.427035	7.690244e-4
1/40, 1/1600	0.013835	0.018186	1.913819e-4
1/80, 1/1600	0.005825	0.008444	4.700194e-5
1/160, 1/1600	0.002536	0.011451	1.087647e-5
1/1600, 1/20	0.209242e-4	2.242296e-4	1.149126e-2
1/1600, 1/40	0.220734e-4	2.259426e-4	2.648148e-3
1/1600, 1/80	0.246042e-4	2.274373e-4	6.070708e-4
1/1600, 1/160	0.245033e-4	2.206108e-4	1.507808e-4

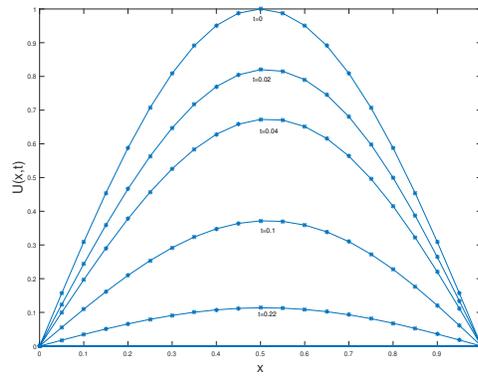
As seen from Table 3, wave amplitudes decrease for the $\nu = 1, 10^{-1}, 10^{-2}, 10^{-3}$ as the time progresses, but for $\nu = 10^{-4}, 10^{-5}$ amplitudes increase until $t = 0.6$ and then decrease. Moreover, the error norm L_∞ were compared with the Ref. [13, 16] for different h and Δt in Table 4. It can be seen from the table that the error norms of L_∞ calculated by the presented method are lower than those given in other Refs.

Problem 2

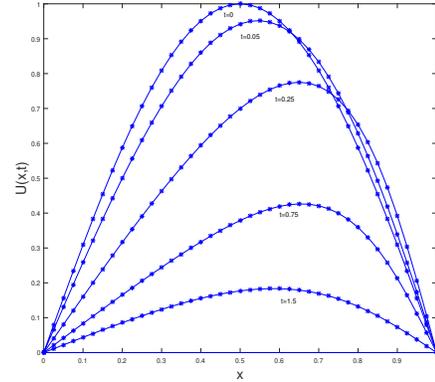
As a second problem, the exact solution of the Burgers' equation giving the shock wave solution is as follows

$$U(x, t) = \frac{x/t}{1 + \sqrt{t/t_0}e^{x^2/4\nu t}}, \quad t \geq 1, 0 \leq x \leq 1.$$

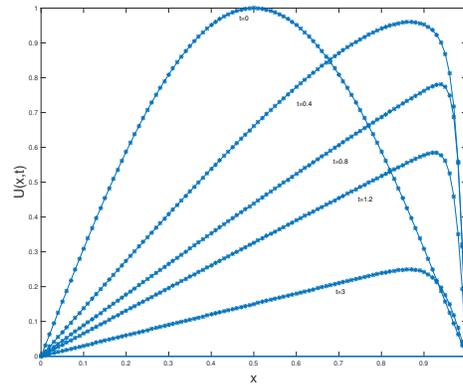
Where $t_0 = e^{1/8\nu}$, boundary conditions were selected from (2) and (3) and the initial condition was obtained by taking $t = 1$ in the exact solution. For this problem, numerical results and graphs are given for $h = 0.02, 0.005, \nu = 0.01, 0.005, 0.0005$ and $\Delta t = 0.01$, which are frequently used in the literature. In Table 5, L_2 and L_∞ are given and compared with some studies in the literature. It is seen that from the Table, the error norms obtained by the method suggested are better than the other studies. In the Table 6, the waves' maximum amplitudes and positions for different h, ν , and t are given. As shown from the table, as time progresses, the waves' peak moves to the right, and their amplitudes decrease. In Figure 2, the error graphs in



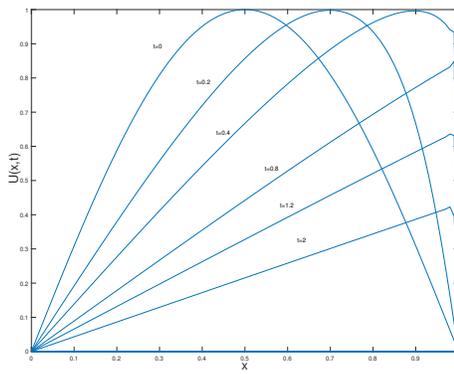
(a) $h = 0.05, \Delta t = 0.01, \nu = 1$



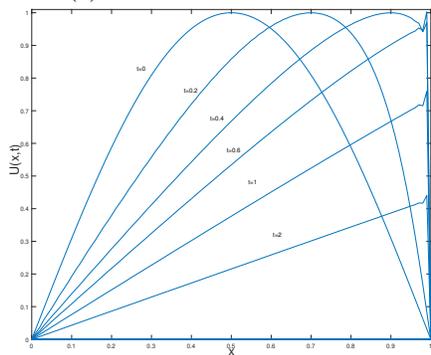
(b) $h = \Delta t = 0.025, \nu = 0.1$



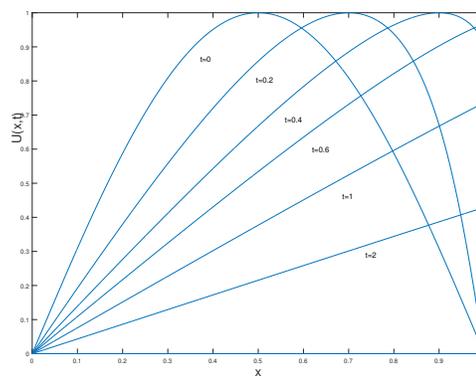
(c) $h = \Delta t = \nu = 10^{-2}$



(d) $h = \Delta t = 10^{-2}, \nu = 10^{-3}$



(e) $h = \Delta t = 10^{-2}, \nu = 10^{-4}$



(f) $h = 10^{-3}, \Delta t = 10^{-2}, \nu = 5 \times 10^{-3}$

Figure 1: Solution profile of Problem 1 for different viscosity and time values.

Table 5: Comparison of the error norms of L_2 and L_∞ with some studies for $h = 0.02$, 0.005 , $\Delta t = 0.01$ of Problem 2.

	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
$h = 0.02, \nu = 0.01$	$t = 1.7$		$t = 2.1$		$t = 2.6$	
Present	0.138431	0.359225	0.172995	0.724110	1.085721	5.608637
Present $x \in [0, 1.2]$	0.138127	0.359225	0.113566	0.288767	0.096592	0.228696
[23] QBCM1	0.17014	0.40431	0.20476	0.86363	1.29951	6.69425
[23] QBCM2-split	0.24003	0.48800	0.30849	1.14760	1.57548	8.06799
[46] Septic col.	0.69910	3.13476	0.72976	2.66986	1.74570	8.06798
$h = 0.02, \nu = 0.005$	$t = 1.8$		$t = 2.4$		$t = 3.2$	
Present	0.183206	0.535872	0.137983	0.388240	0.690508	4.095068
Present $x \in [0, 1.2]$	0.183206	0.535872	0.137909	0.388240	0.106842	0.278624
[23] QBCM1	0.19127	0.54058	0.14246	0.39241	0.93617	5.54899
[23] QBCM2- split	0.49130	1.16930	0.41864	0.93664	1.28863	7.49147
[46] Septic col.	0.68761	2.47189	0.67943	2.16784	1.48559	7.49146
$h = 0.005, \nu = 0.005$	$t = 1.7$		$t = 2.4$		$t = 3.1$	
Present	0.023510	0.079707	0.015469	0.052714	0.556090	4.103998
Present $x \in [0, 1.2]$	0.023510	0.079707	0.014079	0.049002	0.009736	0.033877
[23] QBCM1	0.01705	0.06192	0.01252	0.05882	0.60199	4.43469
[23] QBCM2- split	0.35891	1.21170	0.25132	0.80777	0.63052	4.79061
[23] QBCM1 $x \in [0, 1.2]$	–	–	–	–	0.00765	0.01831
[23] QBCM2 $x \in [0, 1.2]$	–	–	–	–	4.79061	0.583
[40] QBGM	0.35133	1.20755	0.24451	0.80187	0.63335	4.79061
[40] CBGM	0.35126	1.20726	0.24448	0.80176	0.63340	4.79061
[47] Galerkin	0.857	2.576	0.423	1.242	0.235	0.688

$t = 3.1$ are given in addition to the numerical and complete solution for $\nu = 0.005$, 0.0005 , $\Delta t = 0.01$ and $h = 0.005$. As it seen from the graph, while the error in the range $[0, 1]$ for $\nu = 0.005$ is at the right limit, the error decreases and shifts to the left when the range is extended to $[0, 1.2]$.

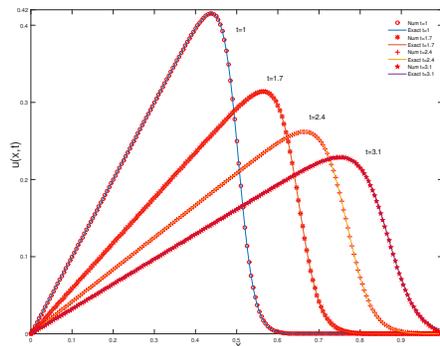
Problem 3

As the last problem, we will consider the Burgers’ equation with the following analytical solution is given by [48, 49] and boundary conditions

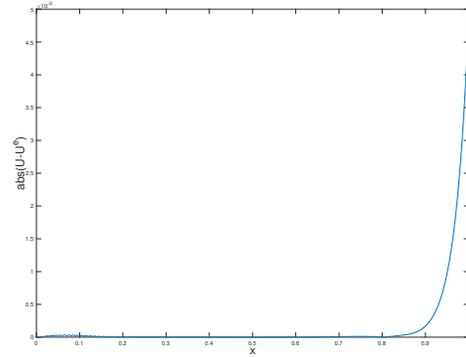
$$U(x, t) = \frac{\alpha + \mu + (\alpha - \mu)e^\eta}{1 + e^\eta}, \eta = \frac{\alpha(x - \mu t - \gamma)}{\nu}, 0 \leq x \leq 1, t \geq 0,$$

$$U(0, t) = 1 \quad \text{and} \quad U(1, t) = 0.2.$$

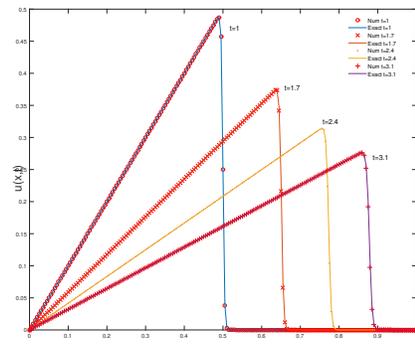
Where μ represents the velocity of the wave moving to the right, while $\alpha = 0.4$, $\mu = 0.6$, and $\gamma = 0.125$ are constants. The initial condition of the problem can be obtained by taking $t = 0$ in the exact solution. Numerical solutions have been obtained up to $t = 1.5$ using $h = 1/36$, $\Delta t = 0.01$, $\nu = 0.01$ parameter values to compare with the studies given in the literature. In the Table 7, L_2 and L_∞ error



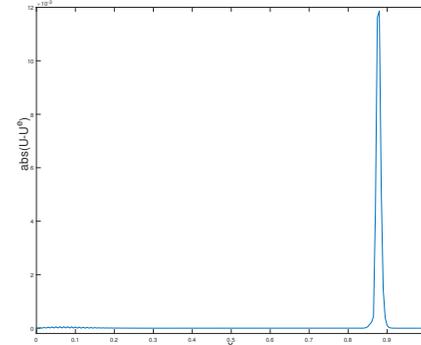
(a) $\nu = 0.005$



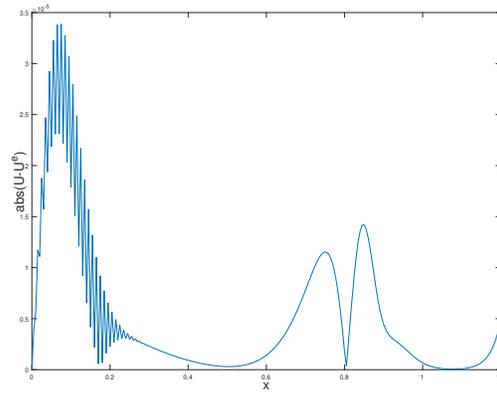
(b) Error at $t = 3.1$ for $\nu = 0.005$



(c) $\nu = 0.0005$



(d) Error at $t = 3.1$ for $\nu = 0.0005$



(e) Error at $t = 3.1$ for $\nu = 0.005$ and $x \in [1.2]$

Figure 2: The shock wave profile of Problem 2 at different times for $h = 0.005$, $\Delta t = 0.01$.

Table 6: Wave positions and maximum amplitudes of Problem 2 for $h = 0.02, 0.005$, $\Delta t = 0.01$.

	t	x	$U(x, t)$	$U^e(x, t)$		t	x	$U(x, t)$	$U^e(x, t)$
$h = 0.02$ $\nu = 0.01$	1	0.420	0.362438	0.362438	$h = 0.02$ $\nu = 0.005$	1	0.440	0.415249	0.415249
	1.7	0.520	0.269753	0.269683		1.8	0.580	0.304811	0.304796
	2.1	0.580	0.239441	0.239444		2.4	0.660	0.261816	0.261799
	2.6	0.640	0.212303	0.212237		3.2	0.760	0.225061	0.225036
$\nu = 0.005$ $h = 0.005$	1	0.440	0.415249	0.415249	$h = 0.005$ $\nu = 0.0005$	1	0.490	0.486554	0.486554
	1.7	0.565	0.314147	0.314106		1.7	0.635	0.374055	0.372725
	2.4	0.665	0.261942	0.261921		2.4	0.755	0.314174	0.313640
	3.1	0.750	0.228863	0.228852		3.1	0.860	0.276033	0.275810

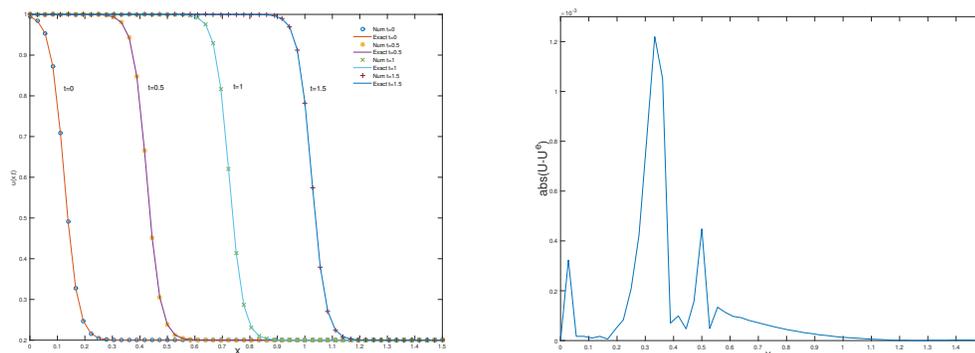
Table 7: Comparison of the error norms L_2 and L_∞ of the Problem 3 with Ref. [23, 40] for $h = 1/36$, $\Delta t = 0.01$, $\nu = 0.01$, $t = 0.5$.

		[23]	[23]	[40]	[40]
	Present	QBCM1	QBCM2	QBGM	CBGM
$L_2 \times 10^3$	0.330303	0.77033	1.81958	1.92558	1.73106
$L_\infty \times 10^3$	1.220225	3.03817	6.94015	6.35489	5.48892

norms are given at time $t = 0.5$ and Compared to the quartic B-spline collocation [23] and [40] using both quadratic and cubic Galerkin method. It can be seen from the Table that the method we recommend has smaller L_2 and L_∞ error norms. In Figure 3, the waves appear to move to the right, and Since the wave does not fit in the range $[0, 1]$ at $t = 1.5$, the solution region has been expanded to $[0, 1.5]$.

6. CONCLUSIONS

As a result of this study, the method proposed here has been found to be effective and reliable in obtaining numerical solutions of partial differential equations. In the [23] study, the collocation method with quartic B-spline bases was applied to the Burgers' equation by first-order splitting (QBCM2). We have achieved better results by applying the second-order Strang splitting method to the Burgers' equation using the same base and method. Since simpler sub-schemes are obtained with the proposed method, this method will be useful in the numerical solution of many one-dimensional or higher-dimensional nonlinear partial differential equations. Besides, the method is noteworthy for numerical solutions and for the solution of complex nonlinear partial differential equations that model various physical phenomena.



(a) Numerical and exact solution for $\nu = 0.01$

(b) Error at $t = 0.5$

Figure 3: The profile of Problem 3 at different times for $h = 1/36$, $\Delta t = 0.01$.

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