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BI-PERIODIC JACOBSTHAL LUCAS MATRIX SEQUENCE

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Abstract. In this paper, the bi-periodic Jacobsthal Lucas sequence will be carried to matrix algebra. The terms of the bi-periodic Jacobsthal Lucas matrix sequence are the bi-periodic Jacobsthal Lucas numbers. By studying the properties of this matrix sequence, the well-known Simpson's formula, generating function as well as the Binet formula are investigated. Some new properties and two summation formulas for this new generalized matrix sequence are also obtained. 2010

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1. Introduction

The increasing applications of integer sequences such as Fibonacci, Lucas, Jacobsthal, Jacobsthal Lucas, Pell, etc in the various fields of science and arts can not be overemphasized. For example, the ratio of two consecutive Fibonacci numbers converges to what is widely known as the Golden ratio whose applications appear in many research areas, particularly in Engineering, Physics, Architecture, Nature, and Art.

The same can easily be said for Jacobsthal sequence. For instance, it is known that microcontrollers and other computers change the flow of execution of a program using conditional instructions. Along with branch instructions, some microcontrollers use skip instructions which conditionally bypass the next instruction which boil down to being useful for one case out of the four possibilities on 2 bits, 3 cases on 3 bits, 5 cases on 4 bits, 11 on 5 bits, 21 on 6 bits, 43 on 7 bits, 85 on 8 bits and continue in that order, which is exactly the Jacobsthal numbers.

Now, the classical Jacobsthal sequence $\{\hat{\jmath}_n\}_{n=0}^{\infty}$ which was named after the German mathematician Ernst Jacobsthal is defined recursively by the relation $\hat{\jmath}_n = \hat{\jmath}_{n-1} + 2\hat{\jmath}_{n-2}$ with initial conditions $\hat{\jmath}_0 = 0$, $\hat{\jmath}_1 = 1$. The other related sequence is the Jacobsthal Lucas sequence $\{\hat{c}_n\}_{n=0}^{\infty}$ which satisfies the same recurrence relation, that is $\hat{c}_n = \hat{c}_{n-1} + 2\hat{c}_{n-2}$ but with different initial conditions $\hat{c}_0 = 2$, $\hat{c}_1 = 1$ in [1].

In [2, 3, 4], the authors defined the bi-periodic Fibonacci sequence and investigated its properties of it in detail. In [12], the authors studied on the convolutions of the bi-periodic Fibonacci numbers. The authors gave a new type of (s, t)-Jacobsthal sequence and defined a binomial form of this sequence in [15]. Bilgici gave identities for the bi-periodic Lucas sequence in [5]. The authors denoted some relations about the bi-periodic Fibonacci sequence by using a special matrix in [7].

Uygun, Owusu in [6] defined a new generalization of Jacobsthal numbers in the following:

$$\begin{aligned} j_n &= \begin{cases} aj_{n-1} + 2j_{n-2}, & \text{if } n \text{ is even} \\ bj_{n-1} + 2j_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2 \qquad (1) \\ &= \left(\frac{b}{a}\right)^{\varepsilon(n)} aj_{n-1} + 2j_{n-2}. \end{aligned}$$

with initial conditions $j_0 = 0,\ j_1 = 1$.The authors investigated some relations about the bi-periodic Jacobsthal sequence in [8]. Gul studied on the bi-periodic Jacobsthal and Jacobsthal-Lucas quaternions in [10]. The authors defined the bi-periodic Pell-Lucas sequence in [11]. In [13], the authors also brought into light the bi-periodic Jacobsthal Lucas sequence $\{c_n\}_{n=0}^{\infty}$ as

$$\begin{aligned} c_n &= \begin{cases} bc_{n-1} + 2c_{n-2}, & \text{if } n \text{ is even} \\ ac_{n-1} + 2c_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2 \qquad (2) \\ &= \left(\frac{a}{b}\right)^{\varepsilon(n)} bc_{n-1} + 2c_{n-2}. \end{aligned}$$

with initial conditions $c_0 = 2,\ c_1 = a$. From the above definition we obtain the nonlinear quadratic equation for the bi-periodic Jacobsthal Lucas sequence as

$$x^2 - abx - 2ab = 0$$

with roots α and β defined by

$$\alpha = \frac{ab + \sqrt{a^2b^2 + 8ab}}{2}, \quad \beta = \frac{ab - \sqrt{a^2b^2 + 8ab}}{2}. \qquad (3)$$

In [9] Coskun, Taskara carried the bi-periodic Fibonacci and Lucas sequences into the matrix theory and defined bi-periodic Fibonacci and bi-periodic Lucas matrix sequences. Similarly in [14], the bi-periodic Jacobsthal matrix sequence is defined and investigated some basic properties of it.

In this paper, we define the matrix representation of the bi-periodic Jacobsthal Lucas sequence, which we shall call the bi-periodic Jacobsthal Lucas matrix sequence. We then proceed to obtain the nth general term of this new matrix sequence. By studying the algebraic properties of this new matrix sequence, the well-known Cassini or Simpson's formula is obtained. The generating function together with the Binet formula and some summation formulas for this new generalized matrix sequence are investigated.

2. Preliminaries

The bi-periodic Jacobsthal Lucas sequence $\{c_n\}_{n=0}^{\infty}$satisfies the following properties;

- $c_{2n} = (ab+4)c_{2n-2} - 4c_{2n-4,}$
- $c_{2n+1} = (ab+4)c_{2n-1} - 4c_{2n-3}$,
- $c_{n+1} + 2c_{n-1} = (ab+8)j_n$
- $j_{n+1} + 2j_{n-1} = c_n$

α and β defined by (3) satisfy the following properties;

- $(\alpha+2)(\beta+2) = 4$,
- $\alpha + \beta = ab, \qquad \alpha\beta = -2ab$,
- $\beta + 2 = \frac{\beta^2}{ab}, \qquad \alpha + 2 = \frac{\alpha^2}{ab}$,
- $-(\alpha+2)\beta = 2\alpha, \qquad -(\beta+2)\alpha = 2\beta$.

Definition 1. *The bi-periodic Jacobsthal matrix sequence $\{J_n(a,b)\}_{n=0}^{\infty}$ is defined recursively by*

$$J_n(a,b) = \begin{cases} aJ_{n-1}(a,b) + 2J_{n-2}(a,b), & \text{if } n \text{ is even} \\ bJ_{n-1}(a,b) + 2J_{n-2}(a,b), & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2, \tag{4}$$

with the initial conditions given as

$$J_0(a,b) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; J_1(a,b) = \begin{pmatrix} b & 2\frac{b}{a} \\ 1 & 0 \end{pmatrix}.$$

The Binet formula for the bi-periodic Jacobsthal matrix sequence is given

$$J_n = A(\alpha^n - \beta^n) + B\left(\alpha^{2\lfloor \frac{n}{2} \rfloor + 2} - \beta^{2\lfloor \frac{n}{2} \rfloor + 2}\right). \tag{5}$$

where

$$\begin{aligned} A &= \frac{(J_1 - bJ_0)^{\varepsilon(n)}\,(aJ_1 - 2J_0 - abJ_0)^{1-\varepsilon(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}\,(\alpha - \beta)} \quad \textit{and} \\ B &= \frac{b^{\varepsilon(n)} J_0}{(ab)^{\lfloor \frac{n}{2} \rfloor + 1}\,(\alpha - \beta)} \end{aligned}$$

[14].

3. Main Results

For any two non-zero real numbers a and b and any positive integer n, the bi-periodic Jacobsthal Lucas matrix sequence denoted by $\{C_n(a,b)\}_{n=0}^{\infty}$ is defined recursively by

$$C_n(a,b) = \begin{cases} bC_{n-1}(a,b) + 2C_{n-2}(a,b), & \text{if } n \text{ is even} \\ aC_{n-1}(a,b) + 2C_{n-2}(a,b), & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2, \tag{6}$$

with the initial conditions given as

$$C_0(a,b) = \begin{pmatrix} a & 4 \\ 2\frac{a}{b} & -a \end{pmatrix}, \; C_1(a,b) = \begin{pmatrix} a^2 + 4\frac{a}{b} & 2a \\ \frac{a^2}{b} & 4\frac{a}{b} \end{pmatrix}.$$

After then we shall use c_n in place of $c_n(a,b)$ and C_n in place of $C_n(a,b)$.

Theorem 1. *For any integer $n \geq 0$, the nth bi-periodic Jacobsthal Lucas matrix sequence is obtained by using the elements of the bi-periodic Jacobsthal Lucas numbers as*

$$C_n = \begin{pmatrix} \left(\frac{a}{b}\right)^{\varepsilon(n)} c_{n+1} & 2c_n \\ \frac{a}{b} c_n & 2\left(\frac{a}{b}\right)^{\varepsilon(n)} c_{n-1} \end{pmatrix}. \tag{7}$$

Proof. We obtain the proof employing mathematical induction. We will start by noting from (2) that $c_0 = 2,\ c_1 = a, c_{-1} = \frac{-a}{2}$, and $c_2 = ab + 4$. Hence the induction for $n = 0$ and $n = 1$ are satisfied. We now assume that the assertion is true for $n = k$, where k is a positive integer, that is;

$$C_k = \begin{pmatrix} \left(\frac{a}{b}\right)^{\varepsilon(k)} c_{k+1} & 2c_k \\ \frac{a}{b} c_k & 2\left(\frac{a}{b}\right)^{\varepsilon(k)} c_{k-1} \end{pmatrix}.$$

We will end the proof by showing that the equation also holds for $n = k+1$; that is

$$\begin{aligned}
C_{k+1} &= \begin{cases} bC_k + 2C_{k-1}, & \text{if } k+1 \text{ is even} \\ aC_k + 2C_{k-1}, & \text{if } k+1 \text{ is odd} \end{cases} \\
&= b^{\varepsilon(k)} a^{1-\varepsilon(k)}\ C_k + 2\ C_{k-1} \\
&= b^{\varepsilon(k)} a^{1-\varepsilon(k)} \begin{pmatrix} \left(\frac{a}{b}\right)^{\varepsilon(k)} c_{k+1} & 2c_k \\ \frac{a}{b} c_k & 2\left(\frac{a}{b}\right)^{\varepsilon(k)} c_{k-1} \end{pmatrix} \\
&\quad +2 \begin{pmatrix} \left(\frac{a}{b}\right)^{\varepsilon(k)} c_k & 2c_{k-1} \\ \frac{a}{b} c_{k-1} & 2\left(\frac{a}{b}\right)^{\varepsilon(k)} c_{k-2} \end{pmatrix} \\
&= \begin{cases} \begin{pmatrix} ac_{k+1} + 2c_k & 2bc_k + 4c_{k-1} \\ ac_k + 2\frac{a}{b}c_{k-1} & 2ac_{k-1} + 4\frac{a}{b}c_{k-2} \end{pmatrix} & k+1 \text{ even} \\ \qquad = \begin{pmatrix} c_{k+2} & 2c_{k+1} \\ \frac{a}{b}c_{k+1} & 2\frac{a}{b}c_k \end{pmatrix} & k+1 \text{ even} \\ \begin{pmatrix} ac_{k+1} + 2\frac{a}{b}c_k & 2ac_k + 4c_{k-1} \\ \frac{a^2}{b}c_k + 2\frac{a}{b}c_{k-1} & 2ac_{k-1} + 4\frac{a}{b}c_{k-2} \end{pmatrix} & \\ \qquad = \begin{pmatrix} c_{k+2} & 2c_{k+1} \\ \frac{a}{b}c_{k+1} & 2\frac{a}{b}c_k \end{pmatrix} & k+1 \text{ odd} \end{cases} \\
&= \begin{pmatrix} \left(\frac{a}{b}\right)^{\varepsilon(k+1)} c_{k+2} & 2c_{k+1} \\ \frac{a}{b} c_{k+1} & 2\left(\frac{a}{b}\right)^{\varepsilon(k+1)} c_k \end{pmatrix} \quad k+1 \text{ odd}
\end{aligned}$$

Lemma 2. *For any integer $m \geq 0$, the following relations are satisfied:*

$$\begin{aligned}
C_{2m} &= (ab+4)C_{2m-2} - 4C_{2m-4}, \\
C_{2m+1} &= (ab+4)C_{2m-1} - 4C_{2m-3}.
\end{aligned}$$

Proof. The proof is obtained by the definition of the bi-periodic Jacobsthal Lucas matrix sequence.

Theorem 3. *The relations between the bi-periodic Jacobsthal matrix sequence and the bi-periodic Jacobsthal Lucas matrix sequence are given as*

$$\begin{aligned}
C_{n+1} + 2C_{n-1} &= \frac{a}{b}(ab+8)J_n \\
J_{n+1} + 2J_{n-1} &= \frac{b}{a}C_n
\end{aligned}$$

Proof. By taking care of the properties $\varepsilon(n+1)+\varepsilon(n)=1,\ c_{n+1}+2c_{n-1}=(ab+8)j_n$, and $j_{n+1}+2j_{n-1}=c_n$, it is obtained that

$$\begin{aligned}
C_{n+1}+2C_{n-1} &= \begin{pmatrix} \left(\frac{a}{b}\right)^{\varepsilon(n+1)} c_{n+2} & 2c_{n+1} \\ \frac{a}{b}c_{n+1} & 2\left(\frac{a}{b}\right)^{\varepsilon(n+1)} c_n \end{pmatrix} \\
&\quad +2\begin{pmatrix} \left(\frac{a}{b}\right)^{\varepsilon(n-1)} c_n & 2c_{n-1} \\ \frac{a}{b}c_{n-1} & 2\left(\frac{a}{b}\right)^{\varepsilon(n-1)} c_{n-2} \end{pmatrix} \\
&= \begin{pmatrix} \left(\frac{a}{b}\right)^{\varepsilon(n+1)} (c_{n+2}+2c_n) & 2(c_{n+1}+2c_{n-1}) \\ \frac{a}{b}(c_{n+1}+2c_{n-1}) & 2\left(\frac{a}{b}\right)^{\varepsilon(n+1)} (c_n+2c_{n-2}) \end{pmatrix} \\
&= \frac{a}{b}(ab+8)\begin{pmatrix} \left(\frac{a}{b}\right)^{-\varepsilon(n)} j_{n+1} & 2\frac{b}{a}j_n \\ j_n & 2\left(\frac{a}{b}\right)^{-\varepsilon(n)} j_{n-1} \end{pmatrix} \\
&= \frac{a}{b}(ab+8)\begin{pmatrix} \left(\frac{b}{a}\right)^{\varepsilon(n)} j_{n+1} & 2\frac{b}{a}j_n \\ j_n & 2\left(\frac{b}{a}\right)^{\varepsilon(n)} j_{n-1} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
J_{n+1}+2J_{n-1} &= \begin{pmatrix} \left(\frac{b}{a}\right)^{\varepsilon(n+1)} j_{n+2} & 2\frac{b}{a}j_{n+1} \\ j_{n+1} & 2\left(\frac{b}{a}\right)^{\varepsilon(n+1)} j_n \end{pmatrix} \\
&\quad +2\begin{pmatrix} \left(\frac{b}{a}\right)^{\varepsilon(n-1)} j_n & 2\frac{b}{a}j_{n-1} \\ j_{n-1} & 2\left(\frac{b}{a}\right)^{\varepsilon(n-1)} j_{n-2} \end{pmatrix} \\
&= \frac{b}{a}\begin{pmatrix} \left(\frac{a}{b}\right)^{\varepsilon(n)} c_{n+1} & 2c_n \\ \frac{a}{b}c_n & 2\left(\frac{a}{b}\right)^{\varepsilon(n)} c_{n-1} \end{pmatrix} = \frac{b}{a}C_n
\end{aligned}$$

Theorem 4. *The relations between bi-periodic Jacobsthal matrix sequence and bi-periodic Jacobsthal Lucas matrix sequence are given as*

$$\begin{aligned}
i)\ C_0J_n &= J_nC_0 = \left(\frac{b}{a}\right)^{\varepsilon(n)} C_n = \left(\frac{a}{b}\right)^{\varepsilon(n+1)} (J_{n+1}+2J_{n-1}) \\
ii)\ C_nJ_1 &= J_1C_n = \left(\frac{b}{a}\right)^{\varepsilon(n+1)} C_n = \left(\frac{a}{b}\right)^{\varepsilon(n)} (J_{n+2}+2J_n)
\end{aligned}$$

Proof. The results are obtained by the product of matrices and by the relations $j_n = \left(\frac{b}{a}\right)^{\varepsilon(n)} aj_{n-1}+2j_{n-2}$, $c_n = \left(\frac{a}{b}\right)^{\varepsilon(n)} bc_{n-1}+2c_{n-2}$, and $c_n = \left(\frac{a}{b}\right)^{\varepsilon(n)} bj_n+4j_{n-1}$.

Theorem 5. *For any positive integer n, we have*

$$\begin{aligned}\det\left[C_n\right] &= \left\{\begin{array}{ll} -(-2)^n(ab+8)\left(\frac{a}{b}\right) & \text{if } n \text{ is even} \\ -(-2)^n(ab+8)\left(\frac{a}{b}\right)^2 & \text{if } n \text{ is odd}\end{array}\right\} \\ &= -(-2)^n(ab+8)\left(\frac{a}{b}\right)^{1+\varepsilon(n)}\end{aligned}$$

Proof.

$$\det\left[C_0\right] = \det\begin{pmatrix} a & 4 \\ 2\frac{a}{b} & -a\end{pmatrix} = -(ab+8)\left(\frac{a}{b}\right)$$

$$\det\left[C_1\right] = \det\begin{pmatrix} a^2+4\frac{a}{b} & 2a \\ \frac{a^2}{b} & 4\frac{a}{b}\end{pmatrix} = 2(ab+8)\left(\frac{a}{b}\right)^2$$

If we generalize the operations, we get the desired result.

Corollary 6 (Sımpson or Cassini Property for the bi-periodic Jacobsthal Lucas sequence)**.**

$$\left(\frac{b}{a}\right)^{2\varepsilon(n)} c_{n+1}c_{n-1} - \frac{b}{a}c_n^2 = (-2)^{n-1}(ab+8)\left(\frac{a}{b}\right)^{1+\varepsilon(n)}.$$

Theorem 7. *The generating function for the bi-periodic Jacobsthal Lucas matrix sequence is given by*

$$\sum_{m=0}^{\infty} C_m x^m = \frac{C_0 + C_1 x + \left[bC_1 - (ab+2)\,C_0\right]x^2 + 2\left[aC_0 - C_1\right]x^3}{1-(ab+4)x^2+4x^4}.$$

which is expressed in component form as

$$\sum_{m=0}^{\infty} C_m x^m = \frac{\begin{pmatrix} a+(a^2+4\frac{a}{b})x+2ax^2-8\frac{a}{b}x^3 & 4+2ax-(2ab+8)x^2+4ax^3 \\ \frac{a}{2b}\left(4+2ax-(2ab+8)x^2+4ax^3\right) & -a+\frac{4ax}{b}+\left(6a+a^2b\right)x^2-\left(2a^2+\frac{8a}{b}\right)x^3\end{pmatrix}}{1-(ab+4)x^2+4x^4}.$$

Proof. We divide the series into two parts

$$C(x) = \sum_{m=0}^{\infty} C_m x^m = \sum_{m=0}^{\infty} C_{2m}x^{2m} + \sum_{m=0}^{\infty} C_{2m+1}x^{2m+1}.$$

We simplify the even part of the above series as follows

$$C_0(x) = \sum_{m=0}^{\infty} C_{2m}x^{2m} = C_0 + C_2x^2 + \sum_{m=2}^{\infty} C_{2m}x^{2m}$$

By multiplying through by $(ab+4)x^2$ and $4x^4$ respectively, we have

$$(ab+4)x^2C_0(x) = (ab+4)C_0x^2 + (ab+4)\sum_{m=2}^{\infty} C_{2m-2}x^{2m}$$

and

$$4x^4C_0(x) = 4\sum_{m=2}^{\infty} C_{2m-4}x^{2m}.$$

Hence it follows that,

$$\begin{aligned}\left[1-(ab+4)x^2+4x^4\right]C_0(x) &= C_0 + C_2x^2 - (ab+4)C_0x^2 \\ &\quad + \sum_{m=2}^{\infty}\left[C_{2m} - (ab+4)C_{2m-2} + 4C_{2m-4}\right]x^{2m}.\end{aligned}$$

By using Lemma 2, we obtained that;

$$C_0(x) = \frac{C_0 + C_2x^2 - (ab+4)C_0x^2}{1-(ab+4)x^2+4x^4}.$$

Similarly, the odd part of the above series is simplified as follows

$$C_1(x) = \sum_{m=0}^{\infty} C_{2m+1}x^{2m+1} = C_1x + C_3x^3 + \sum_{m=2}^{\infty} C_{2m+1}x^{2m+1}$$

By multiplying through by $(ab+4)x^2$ and $4x^4$ respectively, we obtain

$$(ab+4)x^2C_1(x) = (ab+4)C_1x^3 + (ab+4)\sum_{m=2}^{\infty} C_{2m-1}x^{2m+1}.$$

and

$$4x^4C_1(x) = 4\sum_{m=2}^{\infty} C_{2m-3}x^{2m+1}.$$

Hence it follows that,

$$\begin{aligned}&\left[1-(ab+4)x^2+4x^4\right]C_1(x) \\ =\ & C_1x + C_3x^3 - (ab+4)C_1x^3 \\ & + \sum_{m=2}^{\infty}\left[C_{2m+1} - (ab+4)C_{2m-1} + 4C_{2m-3}\right]x^{2m+1}\end{aligned}$$

By using Lemma 2, we obtained that;

$$J_1(x) = \frac{C_1 x + C_3 x^3 - (ab+4)C_1 x^3}{1-(ab+4)x^2+4x^4}.$$

By combining the two results, we have

$$C(x) = \frac{C_0 + C_1 x + (C_2 - (ab+4)C_0)\, x^2 + (C_3 - (ab+4)C_1)\, x^3}{1-(ab+4)x^2+4x^4}.$$

which can be simplified using (6) as

$$C(x) = \frac{C_0 + C_1 x + [bC_1 - (ab+2)\, C_0]\, x^2 + 2\,[aC_0 - C_1]\, x^3}{1-(ab+4)x^2+4x^4}.$$

So, the proof is completed.

Theorem 8 (Binet Formula)**.** *For every $n > 0$ integer, the Binet formula for the bi-periodic Jacobsthal Lucas matrix sequence is given by*

$$C_n = A\left(\alpha^n - \beta^n\right) + B\left(\alpha^{2\left\lfloor \frac{n}{2}\right\rfloor+2} - \beta^{2\left\lfloor \frac{n}{2}\right\rfloor+2}\right). \tag{8}$$

where

$$A = \frac{(C_1 - aC_0)^{\varepsilon(n)}\,(bC_1 - 2C_0 - abC_0)^{1-\varepsilon(n)}}{(ab)^{\left\lfloor \frac{n}{2}\right\rfloor}(\alpha-\beta)} \quad \text{and}$$

$$B = \frac{a^{\varepsilon(n)} C_0}{(ab)^{\left\lfloor \frac{n}{2}\right\rfloor+1}(\alpha-\beta)}$$

Proof.

$$\Phi(x) = C_0 + C_1 x + [bC_1 - (ab+2)\, C_0]\, x^2 + 2\,[aC_0 - C_1]\, x^3$$

Using partial fraction decomposition, we split $C(x)$ as

$$\begin{aligned} C(x) &= \frac{\Phi(x)}{1-(ab+4)x^2+4x^4} \\ &= \frac{1}{4}\left[\frac{Ax+B}{\left(x^2 - \frac{\alpha+2}{4}\right)} + \frac{Cx+D}{\left(x^2 - \frac{\beta+2}{4}\right)}\right] \end{aligned}$$

By solving for the constants A, B, C, and D above, we express $C(x)$ in partial fraction as

$$\begin{aligned} A+C &= 2aC_0-2C_1 \\ B+D &= bC_1-(ab+2)C_0 \\ -A\left(\frac{\beta+2}{4}\right)-C\left(\frac{\alpha+2}{4}\right) &= C_1 \\ -B\left(\frac{\beta+2}{4}\right)-D\left(\frac{\alpha+2}{4}\right) &= C_0 \end{aligned}$$

$$\begin{aligned} A &= \frac{2\alpha\left(aC_0-C_1\right)+4aC_0}{\alpha-\beta}=\frac{2\alpha P+4aC_0}{\alpha-\beta} \\ B &= \frac{\alpha\left(bC_1-2C_0-abC_0\right)+2b(C_1-aC_0)}{\alpha-\beta}=\frac{\alpha Q-2bP}{\alpha-\beta} \\ C &= \frac{2\beta\left(-aC_0+C_1\right)-4aC_0}{\alpha-\beta}=\frac{-2\beta P-4aC_0}{\alpha-\beta} \\ D &= \frac{\beta\left(-bC_1+2C_0+abC_0\right)-2b(C_1-aC_0)}{\alpha-\beta}=\frac{-\beta Q+2bP}{\alpha-\beta} \end{aligned}$$

$$C(x)=\frac{1}{4\left(\alpha-\beta\right)}\left[\frac{\left\{\begin{array}{c} x(2\alpha P+4aC_0) \\ \alpha Q-2bP \end{array}\right\}}{\left(x^2-\frac{\alpha+2}{4}\right)}+\frac{\left\{\begin{array}{c} x\left(-2\beta P-4aC_0\right) \\ -\beta Q+2bP \end{array}\right\}}{\left(x^2-\frac{\beta+2}{4}\right)}\right]$$

The Maclaurin series expansion of the function $\frac{Ax+B}{x^2-C}$ is expressed in the form

$$\frac{Ax+B}{x^2-C}=-\sum_{n=0}^{\infty}AC^{-n-1}x^{2n+1}-\sum_{n=0}^{\infty}BC^{-n-1}x^{2n}$$

Hence $C(x)$ can be expanded and simplified as

$$C(x)=\frac{1}{4\left(\alpha-\beta\right)}\left[\begin{array}{c} -\sum\limits_{n=0}^{\infty}\left(2\alpha P+4aC_0\right)\left(\frac{\alpha+2}{4}\right)^{-n-1}x^{2n+1} \\ -\sum\limits_{n=0}^{\infty}(\alpha Q-2bP)\left(\frac{\alpha+2}{4}\right)^{-n-1}x^{2n} \\ -\sum\limits_{n=0}^{\infty}\left(-2\beta P-4aC_0\right)\left(\frac{\beta+2}{4}\right)^{-n-1}x^{2n+1} \\ -\sum\limits_{n=0}^{\infty}\left(-\beta Q+2bP\right)\left(\frac{\beta+2}{4}\right)^{-n-1}x^{2n}. \end{array}\right]$$

We divide two parts of the equation as $C(x) = C_1(x) + C_2(x)$ according to the even and odd power of x. First of all, we obtain the even part $C_1(x)$

$$C_1(x) = \frac{-1}{4(\alpha-\beta)} \sum_{n=0}^{\infty} \left[\begin{array}{c} (\alpha Q - 2bP) \frac{4^{n+1}}{(\alpha+2)^{n+1}} \\ +(-\beta Q + 2bP) \frac{4^{n+1}}{(\beta+2)^{n+1}} \end{array} \right] x^{2n}$$

which can be simplifed as

$$\frac{-4^n}{(\alpha-\beta)} \sum_{n=0}^{\infty} \left[\frac{(\beta+2)^{n+1}(\alpha Q - 2bP) + (\alpha+2)^{n+1}(-\beta Q + 2bP)}{(\alpha+2)^{n+1}(\beta+2)^{n+1}} \right] x^{2n}$$

From the identity that $(\alpha+2)(\beta+2) = 4$,we have

$$\frac{1}{4(\alpha-\beta)} \sum_{n=0}^{\infty} \left[\left\{ \begin{array}{c} 2\beta Q (\beta+2)^n \\ +2bP(\beta+2)^{n+1} \end{array} \right\} + \left\{ \begin{array}{c} -2\alpha(\alpha+2)^n Q \\ -2bP(\alpha+2)^{n+1} \end{array} \right\} \right] x^{2n}$$

by using the identity $(\alpha+2) = \frac{\alpha^2}{ab}$, we get

$$\frac{1}{2(\alpha-\beta)} \sum_{n=0}^{\infty} \left(\frac{1}{ab}\right)^{n+1} \left\{ \begin{array}{c} Q\left[ab\left(\beta^{2n+1} - \alpha^{2n+1}\right)\right] \\ +bP\left[\beta^{2n+2} - \alpha^{2n+2}\right] \end{array} \right\} x^{2n}$$

Also, making use of the identity $ab = \alpha + \beta$ gives

$$\begin{aligned}
&= \frac{1}{2(\alpha-\beta)} \sum_{n=0}^{\infty} \left(\frac{1}{ab}\right)^{n+1} \left\{ \begin{array}{c} Q(\alpha+\beta)\left(\beta^{2n+1} - \alpha^{2n+1}\right) \\ -bP\left[\alpha^{2n+2} - \beta^{2n+2}\right] \end{array} \right\} x^{2n} \\
&= \frac{1}{2(\alpha-\beta)} \sum_{n=0}^{\infty} \left(\frac{1}{ab}\right)^{n+1} \left\{ [Q(-2ab)(\beta^{2n} - \alpha^{2n})] + (-bP - Q)\left[\alpha^{2n+2} - \beta^{2n+2}\right] \right\} \\
&= \frac{1}{2(\alpha-\beta)} \sum_{n=0}^{\infty} \left(\frac{1}{ab}\right)^{n+1} \left\{ [Q(2ab)(\alpha^{2n} - \beta^{2n})] + 2C_0\left[\alpha^{2n+2} - \beta^{2n+2}\right] \right\} \\
&= \frac{1}{(\alpha-\beta)} \sum_{n=0}^{\infty} \left(\frac{1}{ab}\right)^{n+1} \left\{ \begin{array}{c} [(bC_1 - 2C_0 - abC_0)(ab)(\alpha^{2n} - \beta^{2n})] \\ +C_0\left[\alpha^{2n+2} - \beta^{2n+2}\right] \end{array} \right\}
\end{aligned}$$

In the same way, the odd part of $C(x)$ is $C_2(x)$ obtained as

$$\frac{-4^{n+1}}{4(\alpha-\beta)} \sum_{n=0}^{\infty} \left[+ \frac{\{2\alpha P + 4aC_0\}(\beta+2)^{n+1} + \{-2\beta P - 4aC_0\}(\alpha+2)^{n+1}}{(\alpha+2)^{n+1}(\beta+2)^{n+1}} \right] x^{2n+1}$$

which can be simplified as

$$\frac{-1}{4(\alpha-\beta)}\sum_{n=0}^{\infty}\begin{bmatrix} 2(-2\beta)P(\beta+2)^n+4aC_0(\beta+2)^{n+1} \\ -2(-2\alpha)P(\alpha+2)^n-4aC_0(\alpha+2)^{n+1} \end{bmatrix} x^{2n+1}$$

$\beta+2=-\frac{2\beta}{\alpha},\ \alpha+2=-\frac{2\alpha}{\beta}$ implies gives

$$\frac{-1}{4(\alpha-\beta)}\sum_{n=0}^{\infty}\left\{ \begin{array}{c} 4P\left[\alpha(\alpha+2)^n-\beta(\beta+2)^n\right]+ \\ 4aC_0\left[(\beta+2)^{n+1}-(\alpha+2)^{n+1}\right] \end{array}\right\} x^{2n+1}$$

with $(\alpha+2)=\frac{\alpha^2}{ab}$, we simplify the above expression as

$$\frac{-1}{(\alpha-\beta)}\sum_{n=0}^{\infty}\left(\frac{1}{ab}\right)^{n+1}\left\{ \begin{array}{c} abP(\alpha^{2n+1}-\beta^{2n+1}) \\ -aC_0(\alpha^{2n+2}-\beta^{2n+2}) \end{array}\right\} x^{2n+1}$$

This can be further expanded and simplified as

$$\sum_{n=0}^{\infty}\left\{(C_1-aC_0)\left[\frac{\alpha^{2n+1}-\beta^{2n+1}}{(ab)^n(\alpha-\beta)}\right]+aC_0\left[\frac{\alpha^{2n+2}-\beta^{2n+2}}{(ab)^{n+1}(\alpha-\beta)}\right]\right\} x^{2n+1}$$

Now the even and the odd expressions obtained can be condensed using the parity function $\epsilon(n)$ as

$$C(x)=\sum_{n=0}^{\infty}\left\{ \begin{array}{c} (C_1-aC_0)^{\varepsilon(n)}(bC_1-2C_0-abC_0)^{1-\varepsilon(n)}\left\{\frac{\alpha^n-\beta^n}{(ab)^{\lfloor\frac{n}{2}\rfloor}(\alpha-\beta)}\right\} \\ +a^{\varepsilon(n)}C_0\left\{\frac{\alpha^{2(\lfloor\frac{n}{2}\rfloor+1)}-\beta^{2(\lfloor\frac{n}{2}\rfloor+1)}}{(ab)^{\lfloor\frac{n}{2}\rfloor+1}(\alpha-\beta)}\right\} \end{array}\right\} x^n$$

Therefore compared with $C(x)=\sum_{n=1}^{\infty}C_nx^n$ we obtain the desired result.

Theorem 9 (Alternative Binet Formula)**.** *For every $n>0$ integer, the Binet formula for the bi-periodic Jacobsthal Lucas matrix sequence is given by*

$$C_n=\frac{a^{\varepsilon(n)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor}b}\left\{\left[aJ_1-abJ_0+\alpha J_0\right]\alpha^n+\left[aJ_1-abJ_0+\beta J_0\right]\beta^n\right\}.$$

Proof. Let n even, then

$$\begin{aligned}\frac{b}{a}C_n &= J_{n+1}+2J_{n-1}=\frac{(J_1-bJ_0)\left(\alpha^{n+1}-\beta^{n+1}\right)}{(ab)^{\frac{n}{2}}(\alpha-\beta)}+\frac{bJ_0\left(\alpha^{n+2}-\beta^{n+2}\right)}{(ab)^{\frac{n}{2}+1}(\alpha-\beta)} \\ &\quad +2\frac{(J_1-bJ_0)\left(\alpha^{n-1}-\beta^{n-1}\right)}{(ab)^{\frac{n}{2}-1}(\alpha-\beta)}+2\frac{bJ_0\left(\alpha^n-\beta^n\right)}{(ab)^{\frac{n}{2}}(\alpha-\beta)} \\ &= \frac{\alpha^n}{(ab)^{\frac{n}{2}}}\left[J_1-bJ_0+\frac{\alpha J_0}{a}\right]+\frac{\beta^n}{(ab)^{\frac{n}{2}}}\left[J_1-bJ_0+\frac{\beta J_0}{a}\right]\end{aligned}$$

Let n odd, then

$$\begin{aligned}
\frac{b}{a}C_n &= J_{n+1} + 2J_{n-1} \\
&= \frac{(aJ_1 - 2J_0 - abJ_0)\left(\alpha^{n+1} - \beta^{n+1}\right)}{(ab)^{\frac{n+1}{2}}(\alpha - \beta)} + \frac{J_0\left(\alpha^{n+3} - \beta^{n+3}\right)}{(ab)^{\frac{n+3}{2}}(\alpha - \beta)} \\
&\quad +2\frac{(aJ_1 - 2J_0 - abJ_0)\left(\alpha^{n-1} - \beta^{n-1}\right)}{(ab)^{\frac{n-1}{2}}(\alpha - \beta)} + 2\frac{J_0\left(\alpha^{n+1} - \beta^{n+1}\right)}{(ab)^{\frac{n+1}{2}}(\alpha - \beta)} \\
&= \frac{\alpha^n}{(ab)^{\frac{n+1}{2}}}\left[aJ_1 - abJ_0 + \alpha J_0\right] + \frac{\beta^n}{(ab)^{\frac{n+1}{2}}}\left[aJ_1 - abJ_0 + \beta J_0\right]
\end{aligned}$$

If we combine the results we get the formula.

Theorem 10. *Let $ab \neq 1$, then the summation formula for the bi-periodic of Jacobsthal Lucas matrix sequence is computed as*

$$\sum_{k=0}^{n-1} C_k = \frac{\begin{array}{c} 4\left[(a+1)^{1-\varepsilon(n)}C_{n-2} + 2^{1-\varepsilon(n)}C_{n-3}\right] \\ -\left[(a+1)^{1-\varepsilon(n)}C_n - 2^{1-\varepsilon(n)}C_{n-1}\right] + C_1(b+1) - C_0(1+ab)\end{array}}{1-ab}$$

Proof. Let n even. By using the Binet formula for the bi-periodic of Jacobsthal Lucas matrix sequence, we get

$$\begin{aligned}
\sum_{k=0}^{n-1} C_k &= \sum_{k=0}^{\frac{n-2}{2}} C_{2k} + \sum_{k=0}^{\frac{n-2}{2}} C_{2k+1} \\
&= \sum_{k=0}^{\frac{n-2}{2}} \left\{ \begin{array}{c} \frac{(bC_1 - 2C_0 - abC_0)}{(ab)^k(\alpha-\beta)}\left(\alpha^{2k} - \beta^{2k}\right) \\ + \frac{C_0}{(ab)^{k+1}(\alpha-\beta)}\left(\alpha^{2k+2} - \beta^{2k+2}\right) \end{array} \right\} \\
&\quad + \sum_{k=0}^{\frac{n-2}{2}} \left\{ \begin{array}{c} \frac{(C_1 - aC_0)}{(ab)^k(\alpha-\beta)}\left(\alpha^{2k+1} - \beta^{2k+1}\right) \\ + \frac{aC_0}{(ab)^{k+1}(\alpha-\beta)}\left(\alpha^{2k+2} - \beta^{2k+2}\right) \end{array} \right\}.
\end{aligned}$$

If we use the property of geometric series, we get

$$
\begin{aligned}
= &\left\{\frac{(bC_1-2C_0-abC_0)}{(\alpha-\beta)}\left(\frac{\alpha^n-(ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}-1}(\alpha^2-ab)}-\frac{\beta^n-(ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}-1}(\beta^2-ab)}\right)\right\}\\
&+\left\{\frac{C_0}{(\alpha-\beta)}\left(\frac{\alpha^{n+2}-\alpha^2(ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}}(\alpha^2-ab)}-\frac{\beta^{n+2}-\beta^2(ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}}(\beta^2-ab)}\right)\right\}\\
&+\left\{\frac{(C_1-aC_0)}{(\alpha-\beta)}\left(\frac{\alpha^{n+1}-\alpha(ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}-1}(\alpha^2-ab)}-\frac{\beta^{n+1}-\beta(ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}-1}(\beta^2-ab)}\right)\right\}\\
&+\left\{\frac{aC_0}{(\alpha-\beta)}\left(\frac{\alpha^{n+2}-\alpha^2(ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}}(\alpha^2-ab)}-\frac{\beta^{n+2}-\beta^2(ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}}(\beta^2-ab)}\right)\right\}
\end{aligned}
$$

After some algebraic operations, we have

$$
\begin{aligned}
&\frac{\begin{array}{c}4C_{n-2}+4C_{n-1}-C_n-C_{n+1}-2C_{n-1}\\ +C_1(b+1)+C_0(-1-ab)\end{array}}{1-ab}\\
=\ &\frac{\begin{array}{c}4(a+1)C_{n-2}+8C_{n-3}-(a+1)C_n-2C_{n-1}\\ +C_1(b+1)+C_0(-1-ab)\end{array}}{1-ab}
\end{aligned}
$$

If we make a similar operation for the odd elements of bi-periodic of Jacobsthal Lucas matrix sequence we obtain

$$
\frac{4C_{n-3}+4C_{n-2}-C_n-C_{n-1}+C_1(b+1)+C_0(-1-ab)}{1-ab}
$$

If we combine the results we get the desired result as

$$
\sum_{k=0}^{n-1}C_k=\frac{\begin{array}{c}4\left[(a+1)^{1-\varepsilon(n)}C_{n-2}+2^{1-\varepsilon(n)}C_{n-3}\right]\\ -\left[(a+1)^{1-\varepsilon(n)}C_n+2^{1-\varepsilon(n)}C_{n-1}\right]\\ +C_1(b+1)-C_0(1+ab)\end{array}}{1-ab}
$$

Theorem 11. *Let $ab\neq 1$, then the summation formula for the bi-periodic of Jacobsthal Lucas matrix sequence with a negative power of x is computed as*

$$
\sum_{k=0}^{n-1}\frac{C_k}{x^k}=\frac{1}{(4-(ab+4)x^2+x^4)}\left(\begin{array}{c}\frac{4C_{n-1}}{x^{n-1}}+\frac{4C_{n-2}}{x^{n-2}}-\frac{C_{n+1}}{x^{n-3}}-\frac{C_n}{x^{n-4}}\\ -2(C_1-aC_0)x+(bC_1-2C_0-abC_0)x^2+C_1x^3+C_0x^4\end{array}\right).
$$

Proof. Let n odd. By using Binet formula for bi-periodic of Jacobsthal Lucas matrix sequence, we get

$$\begin{aligned}
\sum_{k=0}^{n-1} C_k &= \sum_{k=0}^{\frac{n-1}{2}} C_{2k} + \sum_{k=0}^{\frac{n-3}{2}} C_{2k+1} \\
&= \sum_{k=0}^{\frac{n-1}{2}} \left\{ \begin{array}{c} \frac{(bC_1-2C_0-abC_0)}{(abx^2)^k(\alpha-\beta)} \left(\alpha^{2k}-\beta^{2k}\right) \\ +\frac{C_0}{(ab)^{k+1}x^{2k}(\alpha-\beta)} \left(\alpha^{2k+2}-\beta^{2k+2}\right) \end{array} \right\} \\
&\quad + \sum_{k=0}^{\frac{n-2}{2}} \left\{ \begin{array}{c} \frac{(C_1-aC_0)}{(ab)^k x^{2k+1}(\alpha-\beta)} \left(\alpha^{2k+1}-\beta^{2k+1}\right) \\ +\frac{aC_0}{(ab)^{k+1}x^{2k+1}(\alpha-\beta)} \left(\alpha^{2k+2}-\beta^{2k+2}\right) \end{array} \right\}.
\end{aligned}$$

If we use the property of geometric series, we get

$$\begin{aligned}
= &\left\{ \frac{(bC_1-2C_0-abC_0)}{(\alpha-\beta)} \left(\frac{\alpha^{n+1}-\left(abx^2\right)^{\frac{n+1}{2}}}{(ab)^{\frac{n-1}{2}}(\alpha^2-abx^2)} - \frac{\beta^{n+1}-(ab)^{\frac{n+1}{2}}}{(ab)^{\frac{n-1}{2}}(\beta^2-abx^2)} \right) \right\} \\
&+ \left\{ \frac{C_0}{(\alpha-\beta)} \left(\frac{\alpha^{n+3}-\alpha^2\left(abx^2\right)^{\frac{n+1}{2}}}{(ab)^{\frac{n}{2}}(\alpha^2-abx^2)} - \frac{\beta^{n+3}-\beta^2\left(abx^2\right)^{\frac{n+1}{2}}}{(ab)^{\frac{n}{2}}(\beta^2-abx^2)} \right) \right\} \\
&+ \left\{ \frac{(C_1-aC_0)}{(\alpha-\beta)} \left(\frac{\alpha^{n}-\alpha\left(abx^2\right)^{\frac{n-1}{2}}}{(abx^2)^{\frac{n-3}{2}}(\alpha^2-abx^2)x} - \frac{\beta^{n}-\beta\left(abx^2\right)^{\frac{n-1}{2}}}{(abx^2)^{\frac{n-3}{2}}(\beta^2-abx^2)x} \right) \right\} \\
&+ \left\{ \frac{aC_0}{(\alpha-\beta)} \left(\frac{\alpha^{n+1}-\alpha^2\left(abx^2\right)^{\frac{n-1}{2}}}{(abx^2)^{\frac{n-1}{2}}x(\alpha^2-abx^2)} - \frac{\beta^{n+1}-\beta^2\left(abx^2\right)^{\frac{n-1}{2}}}{(abx^2)^{\frac{n-1}{2}}x(\beta^2-abx^2)} \right) \right\}
\end{aligned}$$

After some algebraic operations, we have

$$= \frac{(bC_1 - 2C_0 - abC_0)}{(\alpha-\beta)\,(4-(ab+4)x^2+x^4)x^{n-1}\,(ab)^{\frac{n+3}{2}}}\begin{pmatrix} 4a^2b^2(\alpha^{n-1}-\beta^{n-1}) \\ -abx^2(\alpha^{n+1}-\beta^{n+1}) \\ +\left(abx^2\right)^{\frac{n+1}{2}}(\alpha^2-\beta^2) \end{pmatrix}$$
$$+\frac{C_0}{(\alpha-\beta)\,(4-(ab+4)x^2+x^4)x^{n-1}\,(ab)^{\frac{n+5}{2}}}\begin{pmatrix} 4a^2b^2(\alpha^{n+1}-\beta^{n+1}) \\ -abx^2(\alpha^{n+3}-\beta^{n+3}) \\ +\left(abx^2\right)^{\frac{n+3}{2}}(\alpha^2-\beta^2) \end{pmatrix}$$
$$+\frac{C_1 - aC_0}{(\alpha-\beta)\,(4-(ab+4)x^2+x^4)x^{n-2}\,(ab)^{\frac{n+1}{2}}}\begin{pmatrix} 4a^2b^2(\alpha^{n-2}-\beta^{n-2}) \\ -abx^2(\alpha^{n}-\beta^{n}) \\ +\left(abx^2\right)^{\frac{n+1}{2}}(\alpha-\beta)+ \\ \left(abx^2\right)^{\frac{n-1}{2}}(2ab)(\alpha-\beta) \end{pmatrix}$$
$$+\frac{aC_0}{(\alpha-\beta)\,(4-(ab+4)x^2+x^4)x^{n-2}\,(ab)^{\frac{n+3}{2}}}\begin{pmatrix} 4a^2b^2(\alpha^{n-1}-\beta^{n-1}) \\ -abx^2(\alpha^{n+1}-\beta^{n+1}) \\ +\left(abx^2\right)^{\frac{n+1}{2}}(\alpha^2-\beta^2) \end{pmatrix}$$

By using (8), we get

$$\frac{1}{(4-(ab+4)x^2+x^4)}\begin{pmatrix} \frac{4C_{n-1}}{x^{n-1}}+\frac{4C_{n-2}}{x^{n-2}}-\frac{C_{n+1}}{x^{n-3}}-\frac{C_n}{x^{n-4}} \\ -2(C_1-aC_0)x+(bC_1-2C_0-abC_0)x^2+C_1x^3+C_0x^4 \end{pmatrix}$$

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